KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -III

PART-3

Prepared by

Satyajit Mohapatra
(Lecturer)
Department of Basic Sciences and Humanities, KIIT Polytechnic BBSR

CONTENTS

S.No	Chapter Name	Page No
1	Complex Numbers	03-10

CHAPTER-

COMPLEX NUMBERS

In the set of real numbers R, the equation $x^2+9=0$ has no solution. In order to find a solution of this type of equation, the real number set needs some extension. This can be done by introducing a new number called as imaginary number like $\sqrt{-2}, \sqrt{-9}, \sqrt{-16}$ … .

Definition:-

Imaginary Number (i):- The number, whose square is a negative number is called as an imaginary number. It is denoted by *i and* defined by $i=\sqrt{-1}$

<u>Properties of i</u>

$i=\sqrt{-1}$, $i^2=\sqrt{-1}\sqrt{-1}=-1, i^3=-i,\ i^4=1,\cdots\cdots\cdots, i^{4n}=1, i^{4n+1}=i, i^{4n+2}=-1, i^{4n+3}=-i$,for any integer *n*.

Definition:-

Purely Imaginary Number:- If a is any real number, then the number $a\ i$ is called as a purely imaginary number.

Example-

$-2i,\quad 5i,\quad 2.67i,\quad 3.2i,\quad \frac{4}{7}i,\quad etc.$ are called as purely imaginary numbers.

Complex Numbers:

If *a and b* are any two real numbers, then the number *a+ib* is called as a complex number. It is denoted by *z* *i.e* $z=a+ib, a,b\in R$ .The set of all complex numbers is called as Complex number set and it is written as ***C.***

Conjugate of a complex number

If $z=a+ib$, be any complex number, then its conjugate, written as $\bar{z}$ and defined by $\bar{z}=a-ib$

Or The conjugate of a complex number $z=a+ib$ will be obtained by changing the sign before i.

Example:

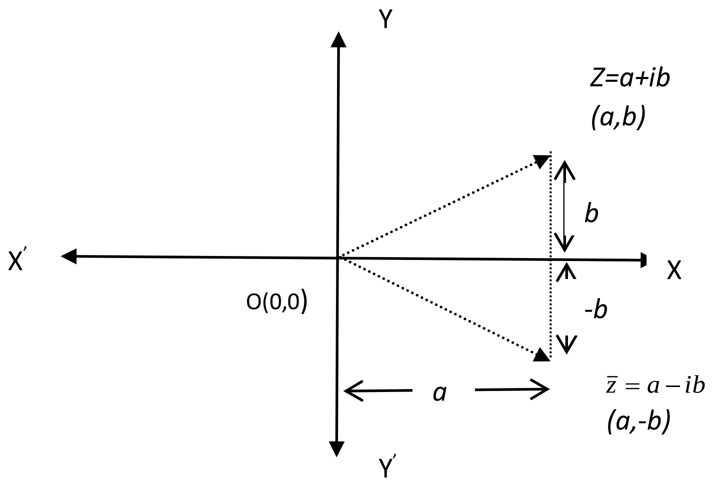
If z=2+9i ,then its conjugate $\bar{z}=2-9i$

If z= - 3 – 5i , then its conjugate $\bar{z}=-3+5i$

If $z=\frac{2}{3}-5i$, then its conjugate $\bar{z}=\frac{2}{3}+5i$

Geometrical Representation Of Complex Numbers:

If $z=a+ib,\ a,b\in R$,then this number corresponds to an ordered pair *(a,b)* in XY-Plane. Hence, any complex number *z* can be represented by a point in the two-dimensional co-ordinate plane. So, the plane representing the complex numbers is called as Complex Plane or ARGAND plane.



Modulus of a complex number

If $z=a+ib$, then its modulus, written as $|z|$ and defined by $|z| = |a+ib| = \sqrt{a^2+b^2}$ = distance between $(0,0)$ *and* (a,b)

Points to remember:

- In $z=a+ib$, *if* $b=0$, then $z=a$ will be a purely real number

  And $a=0$, then $z=ib$ is purely imaginary number.

- In the complex number $z=a+ib$, a is called as real part(written as Re part) and b is called as Imaginary part(written as Im Part).
- Re Part always lies on X –axis. Hence X-axis is called as Real axis.
- Im Part always lies on Y-axis. Hence Y-axis is called as Imaginary Axis.

Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$ and β be any scalar quantity.

Then

1. $z_1 = z_2 \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2$
2. $z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$
3. $z_1 z_2 = (x_1 + iy_1) \times (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = z_2 z_1$
4. $\dfrac{z_1}{z_2} = \dfrac{(x_1 + iy_1)}{(x_2 + iy_2)} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$
5. $\beta z_1 = \beta(x_1 + iy_1) = \beta x_1 + i\beta y_1$

Properties of complex numbers

1. $z_1 + z_2 = z_2 + z_1$
2. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
3. $z_1 z_2 = z_2 z_1$
4. $z_1(z_2 z_3) = (z_1 z_2) z_3$
5. $z_1 \times (z_2 + z_3) = z_1 z_2 + z_1 z_3$
6. $z\bar{z} = |z|^2$

Formulae

1. $(\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$
2. $(\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta) \times (\cos\gamma + i\sin\gamma) = \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$ and so on

CUBE ROOTS OF UNITY $\left(\sqrt[3]{1}\ or\ 1^{\frac{1}{3}}\right)$

Let,

$x = \sqrt[3]{1}\ or\ 1^{\frac{1}{3}}$

$\Rightarrow x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2 + x + 1) = 0$

$\Rightarrow either\ \ x - 1 = 0\ \ or\ \ x^2 + x + 1 = 0$

$\Rightarrow x = 1\ \ or\ \ x = \dfrac{-1 \pm \sqrt{1 - 4\times1\times1}}{2\times1} = \dfrac{-1 \pm i\sqrt{3}}{2}$

Hence, the cube roots of unity are $1, \omega = \dfrac{-1 + i\sqrt{3}}{2},\ \omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$

Properties of the cube roots of unity($1,\ \omega,\ \omega^2$)

1) $1 + \omega + \omega^2 = 0$
2) $\omega^3 = 1$

Questions and answers:

Q.1. Express $\dfrac{-1+2i}{-3+i}$ in the form of $a+ib$

Ans: $\dfrac{-1+2i}{-3+i} = \dfrac{(-1+2i)(-3-i)}{(-3+i)(-3-i)} = \dfrac{3-2i^2 + i(-6+1)}{(-3)^2 - i^2} = \dfrac{3+2-5i}{9+1} = \dfrac{5-5i}{10} = \dfrac{1}{2} - i\dfrac{1}{2}$

Q.2. Express $\dfrac{3-i}{(1+i)^2}$ with rational denominator.

Ans: $\dfrac{3-i}{(1+i)^2} = \dfrac{3-i}{1+i^2+2i} = \dfrac{3-i}{2i} = \dfrac{(3-i)i}{2i^2} = \dfrac{3i-i^2}{-2} = \dfrac{1+3i}{-2} = -\dfrac{1}{2} - i\dfrac{3}{2}$

Q.3. Prove that: $\left(1+\omega^2\right)^4 = \omega$

$1 + \omega + \omega^2 = 0$

Proof: $\Rightarrow 1 + \omega^2 = -\omega$

$\Rightarrow \left(1+\omega^2\right)^4 = (-\omega)^4 = \omega^4 = \omega^3 \times \omega = 1\times\omega = \omega$

Q.4. Prove that $\left(2 + 5\omega + 2\omega^2\right)^6 = 729$

Proof:

$$\left(2+5\omega+2\omega^2\right)^6 = \left(2+2\omega+2\omega^2+3\omega\right)^6$$
$$= \left(2(1+\omega+\omega^2)+3\omega\right)^6$$
$$= \left(2\times0+3\omega^2\right)^6$$
$$= \left(3\omega^2\right)^6 = 3^6 \times \left(\omega^2\right)^6 = 729\times\omega^{12} = 729$$

Assignments:

A. Represent the following in the ARGAND Plane:
   i)1 ii) -3i iii) -2i+1 iv) 3-2i v) 1+2i vi) -2-i
B. Find the square of -3+2i
C. Express $\frac{2i}{1-i}$ in the form a+ib.
D. Find the value of i^{2011}.
E. Express $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$ with rational denominator.
F. Find the product $(3\sqrt{7}\ i-5\sqrt{2}\ i)\times(3\sqrt{2}\ i+5\sqrt{2}\ i)$
G. Prove that: $\left(2+2\omega+5\omega^2\right)^8=729$
H. Prove that: $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$
I. Prove that: $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^2)(1-\omega^5+\omega^4)\ldots\ldots to\ \ 2n\ \ factors=2^{2n}$

DE-MOIVRE'S Theorem:

Statement:

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$,for any integer n.

Proof:

Case-1(If n > 0)

$$(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)\ldots\ldots to\ \ n\ \ factors$$
$$=\cos(\theta+\theta+\ldots n\,times)+i\sin(\theta+\theta+\ldots n\ times)$$
$$=\cos n\theta+i\sin n\theta$$

Case-2(If n < 0)

Let n = - m where m is positive.

$$(\cos\theta+i\sin\theta)^m=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)\ldots\ldots to\ \ m\ \ factors$$
$$=\cos(\theta+\theta+\ldots m\,times)+i\sin(\theta+\theta+\ldots m\ times)$$
$$=\cos m\theta+i\sin m\theta \qquad (By\ \ case-1)$$

$$(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}$$
$$=\frac{1}{(\cos\theta+i\sin\theta)^m}$$
$$=\frac{1}{\cos m\theta+i\sin m\theta}$$
$$=(\cos m\theta+i\sin m\theta)^{-1}$$
$$=\left[(\cos\theta+i\sin\theta)^m\right]^{-1}$$
$$=(\cos\theta+i\sin\theta)^{-m}$$
$$=(\cos\theta+i\sin\theta)^n$$

Square Roots of any Complex Number:

Let us take any complex number $z=a+ib$

To find the square roots of z .i.e. $\sqrt{a+ib}$
Assume,

$$\sqrt{a+ib} = x+iy$$
$$\Rightarrow a+ib = (x+iy)^2$$
$$\Rightarrow a+ib = x^2 - y^2 + i(2xy)$$
$$\Rightarrow a = x^2 - y^2,\ \ b = 2xy$$

Solving for x and y
We have the following:

$$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$$
$$= \sqrt{a^2 + b^2}$$

Now solve :

$$x^2 + y^2 = \sqrt{a^2 + b^2}\, and$$
$$x^2 - y^2 = a$$

to get x and y.

NOTE: To find the value of $\sqrt{a-ib}$ assume $\sqrt{a-ib} = x - iy$

Example:
Find the square roots of 3+4i.

Solution:

Let

$$\sqrt{3+4i} = x+iy$$
$$\Rightarrow 3+4i = x^2 - y^2 + i(2xy)$$
$$\Rightarrow x^2 - y^2 = 3,\ 2xy = 4$$

$$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$$
$$= \sqrt{3^2 + 4^2} = 5$$
$$x^2 - y^2 = 3$$
$$x^2 + y^2 = 5$$

On addition:

$$2x^2 = 8$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

On subtraction:

$$2y^2 = 2$$
$$\Rightarrow y^2 = 1$$
$$\Rightarrow y = \pm 1$$

$\therefore \sqrt{3+4i} = \pm(2+i)$

Example:

Given: $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, show that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$

Proof:

Let, $\cos\alpha + i\sin\alpha = x$, $\cos\beta + i\sin\beta = y$, $\cos\gamma + i\sin\gamma = z$, then

$x + y + z = 0$ (verify!)

Now,

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 0$$

$$\Rightarrow x^3 + y^3 + z^3 - 3xyz = 0$$

$$\Rightarrow x^3 + y^3 + z^3 = 3xyz$$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3 = 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

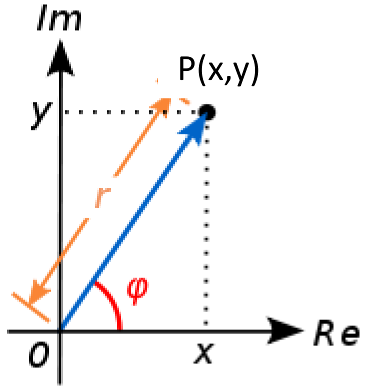
$$\Rightarrow \cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma = 3(\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$$

$$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3(\cos(\alpha+\beta+\gamma))$$

ARGUMENT OF A COMPLEX NUMBER

If $z = x + iy,\ x, y \in R$, then the argument of z ,written as arg(z) and defined by:

$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \varphi$ and geometrically, it is the angle φ made by the line segment joining $(0,0)$ *and* (x, y) with the positive direction of X-axis.



Formulae for the principal argument of a complex number z = x + iy

Find $\tan^{-1}\left|\frac{y}{x}\right|$,use the formulae given below.

Quadrant	Sign of x and y	Arg z
I	x>0, y>0	$\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
II	x<0,y>0	$\pi-\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
III	x<0,y<0	$-\pi+\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
IV	x>0,y<0	$-\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$

By convention, the principal value of the argument satisfies $-\pi < Arg(z) \le \pi$.

Polar Form of a Complex Number

A complex number $z = x + iy$ can be represented by a point P(x,y) as shown in Figure.

The length of the vector $\overrightarrow{OP}$, $r = \left\lvert\overrightarrow{OP}\right\rvert = \sqrt{x^2 + y^2}$, is called the modulus of the complex number z , and it is denoted by $\lvert z\rvert$. Let, $\arg(z) = \varphi = \tan^{-1}\left(\frac{y}{x}\right)$, then the polar form of z will be $z = x + iy = r(\cos\varphi + i\sin\varphi)$

Example: Express 1+i in polar form.

Answer: z=1+i , Here, x=1,y=1.

$$\arg(z) = \varphi = \tan^{-1}\left\lvert\frac{y}{x}\right\rvert = \tan^{-1}\left\lvert\frac{1}{1}\right\rvert = \tan^{-1}1 = \frac{\pi}{4} \text{ and } r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

So, $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Assignments:

1. Find the square roots of the following complex numbers.
   a) $-5+12i$ b) $3-4i$
2. Given: $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0,$ show that
   $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$.
   Hints: Same as example given above.
3. Given: $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0,$ show that
   $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Hints: Same as example given above with

$$(x+y+z)^2 \quad = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = x^2 + y^2 + z^2 + 2xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\Rightarrow x^2 + y^2 + z^2 + 2xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0 \ (Since\ x+y+z=0=x^{-1}+y^{-1}+z^{-1})$$

$$\Rightarrow x^2 + y^2 + z^2 = 0$$

4. If $x+\frac{1}{x}=2\cos\theta$, prove that $x^n+\frac{1}{x^n}=2\cos n\theta$
5. For any positive integer n, show that,

   a) $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}$

   b) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$

6. If $x+\frac{1}{x}=2\cos\alpha,\ x+\frac{1}{x}=2\cos\beta,\ x+\frac{1}{x}=2\cos\gamma$, prove that $xyz+\frac{1}{xyz}=2\cos(\alpha+\beta+\gamma)$

7. Find the argument of the complex numbers:

   a) $1-i\sqrt{3}$ b) $1+i\sqrt{3}$

8. Express $1+i\sqrt{3}$ in polar form.

Reference Books:

Elements of Mathematics Vol. -1 (Odisha State Bureau of Text Book preparation & Production)