KIIT POLYTECHNIC

<u>LECTURE NOTES</u>

ON

ENGG. MATH -III

PART-2

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CONTENTS

CHAPTER-3
MATRICES

A rectangular *arrangement of* mn *numbers with* 'm' *horizontal lines* (*rows*) *and* 'n' *vertical lines* (*columns*) *is known as matrix of order* $m \times n$ *or* (m *by* n).

$$Ex : A = \begin{bmatrix} 3 & 2 & -1 \\ 3 & 5 & 1 \\ 2 & 4 & 3 \end{bmatrix}_{3\times3}$$

Note : An element occuring in the ith row and jth column of a matrix A will be called (i, j)*th element .It is denoted by* a_{ij}.

$$Ex : A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3}$$

It is a matrix of order 2×3 *with general elements.*

<u>*Row matrix*</u>

A matrix having only one row is known as row matrix.

$$Ex : \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}_{1\times3}$$

<u>*Column matrix*</u>

A matrix having only one column is known as column matrix.

$$Ex : \begin{bmatrix} 1 \\ 2 \\ -8 \\ 4 \end{bmatrix}_{4\times1}$$

<u>*Null matrix or Zero matrix*</u>

If all the elements of a matrix are zero then it is called null matrix.It is denoted by $O_{m\times n}$.

$$Ex : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$$

<u>*Square matrix*</u>

If the number of rows and column of a matrix are equal then it is called square matrix.

i.e $m = n$

$$Ex: A=\begin{bmatrix} 1 & 2 & 2 \\ -3 & 4 & -1 \\ 3 & 6 & 1 \end{bmatrix}_{3\times 3}$$

Rectangular matrix

If the number of rows and column of a matrix are not equal then it is called rectangular *matrix.*
i.e $m \neq n$

$$Ex: A=\begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & 1 \end{bmatrix}_{2\times 3}$$

Diagonal matrix

A square matrix is said to be a diagonal matrix if all the diagonal elements are present but non diagonal elements are zero.

$$Ex: A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Scalar matrix

A square matrix is said to be a scalar matrix if all the diagonal elements are same but non diagonal elements are zero.

$$Ex: A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$$

Identity matrix or unit matrix

A square matrix is said to be an unit matrix if all the diagonal elements are unity (1) *but non diagonal elements are zero.It is denoted by* I_n *or* I.

$$Ex: I_3 \, or\ I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad I_2 \, or\ I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Upper Triangular matrix

A square matrix is said to be an upper triangular matrix if all the elements below the main diagonal are zero.

$$Ex: A=\begin{bmatrix} 1 & 2 & 7 \\ 0 & -2 & 6 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$$

Lower Triangular matrix

A square matrix is said to be a lower triangular matrix if all the elements above the main diagonal are zero.

$$Ex: A=\begin{bmatrix}1 & 0 & 0\\ 2 & -2 & 0\\ 5 & 7 & 3\end{bmatrix}_{3\times 3}$$

comparable matrix

Two matrices A and B are said to be comparable if they have same order.

$$Ex: A=\begin{bmatrix}1 & -2 & 3\\ 4 & 1 & 7\end{bmatrix}_{2\times 3} \qquad B=\begin{bmatrix}1 & a & b\\ c & d & 7\end{bmatrix}_{2\times 3}$$

Equal matrices

Two matrices A and B are said to be equal if they have same order and their corresponding elements are equal.

$$A=\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}_{2\times 2} \qquad A=\begin{bmatrix}a_3 & b_3\\ a_4 & b_4\end{bmatrix}_{2\times 2}$$

$A = B$ *iff* $a_1 = a_3, b_1 = b_3, a_2 = a_4, b_2 = b_4$

$Ex:$ *Find the value of x and y if* $\begin{bmatrix}x & 2y\\ 3 & -2\end{bmatrix}=\begin{bmatrix}1 & 4\\ 3 & -2\end{bmatrix}$

$Ans:$ *Given* $\begin{bmatrix}x & 2y\\ 3 & -2\end{bmatrix}=\begin{bmatrix}1 & 4\\ 3 & -2\end{bmatrix}$

$x = 1$, $2y = 4 \Rightarrow y = 2$

Matrix Addition

If A and B are two comparable matrices each of order $m\times n$ *then the addition* $A+B$ *be a matrix obtained by adding the corresponding element of A and B.*

$$Ex: A=\begin{bmatrix}1 & -2 & 3\\ 4 & 1 & 7\end{bmatrix}_{2\times 3} \qquad B=\begin{bmatrix}1 & 5 & 5\\ 2 & 6 & -1\end{bmatrix}_{2\times 3}$$

$$Ans: A+B=\begin{bmatrix}1 & -2 & 3\\ 4 & 1 & 7\end{bmatrix}_{2\times 3}+\begin{bmatrix}1 & 5 & 5\\ 2 & 6 & -1\end{bmatrix}_{2\times 3}=\begin{bmatrix}1+1 & -2+5 & 3+5\\ 4+2 & 1+6 & 7-1\end{bmatrix}=\begin{bmatrix}2 & 3 & 8\\ 6 & 7 & 6\end{bmatrix}_{2\times 3}$$

Scalar multiplication of a matrix

If A be a matrix and k be a scalar then the scalar multiplication of a matrix obtained by multiplying each element of A by k. It is denoted by kA.

$Ex: A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$

$2A = 2\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 6 & -4 \end{bmatrix}$

Transpose of a matrix

If A be a matrix of order $m \times n$ then the matrix obtained by changing the rows and columns then it is called transpose of a matrix. It is denoted by A^T or A' and the order of A^T is $n \times m$.

$Ex: A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$

Matrix multiplication

Existence of the product of two matrices

The product of two matrices A and B will be exist if the number of columns in the 1st matrix (A) is equal to the number of rows in the 2nd matrix (B).

If A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$ then AB be a matrix of order $m \times p$.

Let $A = [\ ]_{m \times n}$ $B = [\ ]_{n \times p}$ $AB = [\ ]_{m \times p}$

$Note: AI = A = IA$

$AA = A^2$

$II = I^2 = I$

Singular matrix

A square matrix 'A' is said to be a singular *matrix if* $\det.A$ *or* $|A| = 0$.

$Ex: A = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

Non – Singular matrix

A square matrix 'A' is said to be a nonsingular *matrix if* $\det.A$ *or* $|A| \neq 0$.

$Ex: A = \begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} = 12 - 8 = 4 \neq 0$

Rank of a matrix

The rank of a matrix is the largest order of any non vanishing minor of the matrix.

OR

The rank of a matrix is the maximum number L.I (Linear indepedent) rows or columns of the matrix.

OR

The number of non zero rows or columns of a triangular matrix is known as rank of a matrix.

Rank of a matrix is denoted by $\rho(A)$ or $r(A)$

How to find the rank of a matrix

1.Let the given matrix is A

2.Before finding rank check the pivot element $(a_{11} \neq 0)$ (i.e the 1st element in the matrix not zero)

3.If the pivot element $(a_{11} = 0)$ then interchange any two rows or columns.

4.Transfer all the elements below the pivot elements to '0' and reduce the matrix to an upper triangular matrix form.

5.After upper triangular matrix(rank of matrix = number of non zero rows or columns)

Note : If we interchange any two rows or columns of a matrix then the rank of a matrix remains unchanged.

Note :

1.If all minor of a matrix of order $(r+1)$ are zero then its rank is $\leq r$.

2.Rank of nth order non singular matrix is n.

3.Rank of nth order singular matrix is $< n$.

4.Rank of nth order square matrix is $\leq n$.

5.Rank of nth order unit matrix is n.

6.Rank of a matrix having order $m \times n$

(a) If $m > n$ then rank is $\leq n$

(b) If $m < n$ then rank is $\leq m$

(c) If $m = n$ then rank is $\leq m = n$

7.The rank of null matrix is zero.

Ex : Find the rank of the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Ans : $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$, the 2nd order minor vanishes.so the rank of the matrix is 1.

$$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$Ans: \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \qquad (Here\ the\ 3rd\ order\ \text{determinant}\ vanishes.)$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$

$$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \left(two\ non\ zero\ rows\ and\ one\ zero\ row\ are\ present\ after\ upper\ triangular\ matrix\right)$$

$Hence\ the\ rank\ of\ the\ matrix\ is\ 2\ \ i.e \qquad r\left(A\right) = 2.$

$$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$$

$$Ans: \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix} \qquad (Here\ the\ 3rd\ order\ \text{determinant}\ vanishes.)$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + \frac{1}{2}R_1$

$$\approx \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(two\ \ zero\ rows\ and\ one\ non\ zero\ row\ are\ present\ after\ simplification\right)$$

$Hence\ the\ rank\ of\ the\ matrix\ is\ 1\ \ i.e \qquad r\left(A\right) = 1.$

$$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$Ans: \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (*Here the* 3*rd order* determinant *vanishes.*)

$R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + R_1$

$$\approx \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{2}R_1$

$$\approx \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the rank of the matrix is 2 *i.e* $r(A) = 2.$

$Ex:$ *Find the rank of the matrix* $\begin{bmatrix} 1 & 5 & 9 \\ 4 & 8 & 12 \\ 7 & 11 & 15 \end{bmatrix}$

$Ex:$ *Find the rank of the matrix* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$Ex:$ *Find the rank of the matrix* $(a)\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix},(b)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix},(c)\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$

$Ex:$ *Find the rank of the matrix* $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$Ans: \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3 - R_2$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Hence\ the\ rank\ of\ the\ matrix\ is\ 3 \quad i.e \quad r(A) = 3.$

$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

$Ans: \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_2, C_4 \to C_4 - C_3$

$$\approx \begin{bmatrix} 5 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 \\ 16 & 1 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2, C_4 \to C_4 - C_3$

$$\approx \begin{bmatrix} 5 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 11 & 1 & 0 & 0 \\ 16 & 1 & 0 & 0 \end{bmatrix}$$

$Hence\ the\ rank\ of\ the\ matrix\ is\ 2 \quad i.e \quad r(A) = 2.$

$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

$$Ex: Find\ the\ rank\ of\ the\ matrix\ (a)\begin{bmatrix} 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 9 \end{bmatrix}, (b)\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}, (c)\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$$

$$Ex: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}_{3\times 4}$$

$$Ans: \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix} \qquad (The\ rank\ is\ either\ 3\ or < 3)$$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - R_1$

$$\approx \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{3}R_2$

$$\approx \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the rank of the matrix is 2 *i.e* $r(A) = 2.$

$$Ex: Find\ the\ rank\ of\ the\ matrix\ (a)\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}, (b)\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} (c)\begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$$

consistent and inconsistent

If the solution of the system of linear equation exist then it is called consitent otherwise it is called inconsistent.

Consistency for system of linear equations

Let us consider the system of 'm' linear equation with 'n' unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n &= d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n &= d_m \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now equation (1) can be written in the form of $AX = B$

$$\text{Where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_m \end{bmatrix}$$

Here A is called coefficient matrix.

$$\text{Let } K = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} & d_1 \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} & d_2 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots & . \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} & d_m \end{array}\right] \text{ which is called augmented matrix}\cdot$$

Rouche's Theorem

The system of equation is consistent iff the rank of A and K are equal otherwise the system of equation is inconsistent.

Note:

(1) *If system of equation is consistent then it gives either unique solution or* infinite *solution. i.e it has at least one solution.*

(2) *If system of equation is inconsistent then it gives no solution.*

procedure for the test of consistency

Let the rank of coefficient matrix $(A) = r$ *and the rank of augmented matrix* $(K) = r'$

Number of variables $= n$

No solution

If $r \neq r'$ *then equations are inconsistent and gives no solution.*

Unique solution

(2) *If* $r = r' = n$ *then equations are consistent and gives unique solution.*

Infinite solution

(3) *If* $r = r' < n$ *then equations are consistent and gives* infinite *solution.*

Ex:

Test the consistency

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

Ans:

The augmented matrix is

$$K = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$\left\{ Here\ A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \right\}$$

$R_1 \leftrightarrow R_2$ (*Two rows are* interchanged)

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 15R$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

here rank of $A = 2$ *and rank of* $K = 2, n = 3$

So it is consistent and gives infinite *solution.*

$Now -6y + 18z = 12 \Rightarrow y - 3z = -2 \Rightarrow y = 3z - 2........(1)$

Again $x + y - 3z = -1 \Rightarrow x - 2 = -1 \Rightarrow x = 1$ *where z is a parameter.*

$Ex:$

$Test\ the\ consistency\quad 2x-3y+7z=5$

$$3x+y-3z=13$$

$$2x+19y-47z=32$$

$Ans:$

$The\ augmented\ matrix\ is$

$$K=\left[\begin{array}{ccc|c}2&-3&7&5\\3&1&-3&13\\2&19&-47&32\end{array}\right]\qquad \left\{Here\ A=\begin{bmatrix}2&-3&7\\3&1&-3\\2&19&-47\end{bmatrix}\right\}$$

$R_2\to 2R_2-3R_1,\ R_3\to R_3-R_1$

$$\approx\left[\begin{array}{ccc|c}2&-3&7&5\\0&11&-27&11\\0&22&-54&27\end{array}\right]$$

$R_3\to R_3-2R_2$

$$\approx\left[\begin{array}{ccc|c}2&-3&7&5\\0&11&-27&11\\0&0&0&5\end{array}\right]\qquad here\ rank\ of\ A=2\ and\ rank\ of\ K=3, n=3$$

$So\ it\ is\ inconsistent\ and\ gives\ no\ solution.$

$Ex:$

$Test\ the\ consistency\quad 2x-3y+z=1$

$$x+2y-3z=4$$

$$4x-y-2z=8$$

$Ans:$

$The\ augmented\ matrix\ is$

$$K=\left[\begin{array}{ccc|c}2&-3&1&1\\1&2&-3&4\\4&-1&-2&8\end{array}\right]\qquad \left\{Here\ A=\begin{bmatrix}2&-3&1\\1&2&-3\\4&-1&-2\end{bmatrix}\right\}$$

$R_2\to 2R_2-R_1,\ R_3\to R_3-2R_1$

$$\approx\left[\begin{array}{ccc|c}2 & -3 & 1 & 1\\0 & 7 & -7 & 7\\0 & 5 & -4 & 6\end{array}\right]$$

$R_3 \to 7R_3 - 5R_2$

$$\approx\left[\begin{array}{ccc|c}2 & -3 & 1 & 1\\0 & 7 & -7 & 7\\0 & 0 & 7 & 7\end{array}\right] \qquad \text{here rank of } A = 3 \text{ and rank of } K = 3, n = 3$$

So it is consistent and gives unique solution.

$Now\ \ 7z = 7 \Rightarrow z = 1$

$7y - 7z = 7 \Rightarrow y - z = 1 \Rightarrow y = 1 + z = 1 + 1 = 2$

$2x - 3y + z = 1 \Rightarrow 2x - 6 + 1 = 1 \Rightarrow 2x = 6 \Rightarrow x = 3$

Ex :

Test the consistency and if possible then find solution

(a)
$$x + 2y - z = 6$$
$$3x - y - 2z = 3$$
$$4x + 3y + z = 9$$

(b)
$$4x + 3y + 2z = -7$$
$$2x + y - 4z = -1$$
$$x + 2y + z = 1$$

(c)
$$2x - 3y + 7z = 5$$
$$5x - 2y + 4z = 18$$
$$2x + 19y - 47z = 32$$

Ex :

Test the consistency and if possible then find solution

$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2$$
$$3x + 9y - z = 4$$

Ans :

The augmented matrix is

$$K = \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\2 & 3 & 2 & 5\\3 & -5 & 5 & 2\\3 & 9 & -1 & 4\end{array}\right] \qquad \left\{ Here\ A = \begin{bmatrix}1 & 2 & 1\\2 & 3 & 2\\3 & -5 & 5\\3 & 9 & -1\end{bmatrix} \right\}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 3R_1$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$$

$R_3 \to R_3 - 11R_2, R_4 \to R_4 + 3R_2$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

$R_4 \to R_4 + 2R_3$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{here rank of } A = 3 \text{ and rank of } K = 3, n = 3$$

So it is consistent and gives unique solution.

Now $2z = 4 \Rightarrow z = 2$

$-y = -1 \Rightarrow y = 1$

$x + 2y + z = 3 \Rightarrow x + 2 + 2 = 3 \Rightarrow x = -1$

Ex:

Test the consistency and if possible then find solution

$$x + 5y + 7z = 15$$
$$2x + 3y + 4z = 11$$
$$x - 2y - 3z = -4$$
$$3x + 11y + 13z = 25$$

$Ex:$ *Investigate for what value of* λ *and* μ *the simultaneous equation*

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has (a) *no solution*, (b) *a unique*, (c) *an infinite number of solution*

$Ans:$ *Here* $A=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$

Now $K=\left[\begin{array}{ccc|c}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right]$

$R_2\to R_2-R_1$, $R_3\to R_3-R_2$

$$\approx\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{array}\right]$$

No solution

If $\lambda=3$ *and* $\mu\neq10$ *then rank of* $A=2$, *rank of* $K=3$

Unique solution

If $\lambda\neq3$ *and* μ *be any value then rank of* $A=$ *rank of* $K=3$

Infinite *solution*

If $\lambda=3$ *and* $\mu=10$ *then rank of* $A=$ *rank of* $K=2<n\,(n=3)$

$Ex:$ *Show that the equations*

$$3x+4y+5z=a$$
$$4x+5y+6z=b$$
$$5x+6y+7z=c$$

do not have a solution unless $a+c=2b$.

$Ans:$ *Here* $A=\begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}a\\b\\c\end{bmatrix}$

Now $K=\left[\begin{array}{ccc|c}3&4&5&a\\4&5&6&b\\5&6&7&c\end{array}\right]$

$R_2\to 3R_2-4R_1$, $R_3\to 3R_3-5R_1$

$$\approx\left[\begin{array}{ccc|c}3 & 4 & 5 & a\\0 & -1 & -2 & 3b-4a\\0 & -2 & -4 & 3c-5a\end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\approx\left[\begin{array}{ccc|c}3 & 4 & 5 & a\\0 & -1 & -2 & 3b-4a\\0 & 0 & 0 & 3c-6b+3a\end{array}\right]$$

Rank of $A = 2$

The system of equation is consistent if the rank of augmented matrix $K = 2$

which is obtained if $3c - 6b + 3a = 0$

$\Rightarrow 3c + 3a = 6b \Rightarrow a + c = 2b$

Hence the equation have no solution unless $a + c = 2b$.

PREVIOUS YEAR QUESTIONS

$Ex-1$: *Find the rank of the matrix* $\begin{bmatrix}1 & 0 & 1\\0 & 2 & 2\\2 & 3 & 4\end{bmatrix}$ (2006)

$Ex-2$: *Test the constiency and hence solve*

$$x+y+z=6$$
$$x-y+2z=5$$
$$3x+y+z=8$$
$$2x-2y+3z=7$$

(2006)

$Ex-3$: *Write down the procedure to test the consistency of a system of equations in 'n' unknowns.* (2007)

$Ex-4$: Determine *the rank of the matrix*

$$\begin{bmatrix}3 & -1 & 2\\-6 & 2 & 4\\-3 & 1 & 2\end{bmatrix}$$

(2007, 2004, 2003)

$Ex-5$: *Define the rank of a matrix.* (2005)

$Ex-6$: Ex: *Investigate for what value of* λ *and* μ *the simultaneous equation*

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has (a) *no solution,* (b) *a unique,* (c) *an* infinite *number of solution* (2005)

$Ex-7: Write\ what\ you\ mean\ null\ matrix.$ (2004)

$Ex-8: Find\ the\ rank\ of\ a\ matrix \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (2002)

$Ex-9: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (2008)

$Ex-10: Investigate\ for\ what\ values\ of\ \lambda\ and\ \mu\ the$ simultaneous equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu \quad have\,(a)\,no\ solution\,(b)\,a\ unique\ solution\,(c)\,an\ \text{infinite}\ number\ of\ solution$$

(2008)

$Ex-11: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 0 \end{bmatrix}$ (2009)

$Ex-13: State\ Rouche's\ theorem.$ (2010)

$Ex-14: Test\ the\ constiency\ and\ solve$

$$5x+3y+7z=4$$
$$3x+26y+2z=9$$
$$7x+2y+10z=5$$

(2010)

$Ex-15: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 2 \end{bmatrix}$ (2011)

$Ex-16: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ (2011)

$Ex-17: Find\ the\ rank\ of\ the\ matrix \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $(2010\ old)$

CHAPTER-3
LAPLACE TRANSFORMS

Def^n: Let f(t) be the real valued function of t, t > 0. Then Laplace transform of f(t) is given by $L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt$,

where s is parameter may be real or complex

Formulas: (Laplace Transformation of some simple functions)

(i) Laplace Transform of constant function :
Let f(t)=k ,where k is constant

Then Laplace transform of f(t) is given by

L{f(t)}=L(k)= $\frac{k}{s}$ *

Ex : L(1)=1/s , L(3)=3/s , L(-3)=-3/s, ……..etc

(ii) Laplace Transform of algebraic function:
Let f(t)=t^n , n=0,1,2,…..

Then Laplace transform of f(t) is given by

L{f(t)}=L(t^n)= $\frac{n!}{s^{n+1}}$, where n! is factorial of n *

n!=n(n-1)(n-2)…..2.1

i. e. 5!=5.4.3.2.1=120 , 4!=4.3.2.1=24 ………etc
Ex: L(t)=1/s^2 , L(t^2)=2/s^3 ,………

(iii) Laplace Transform of exponential function:
Let f(t)=e^{at} ,where a is constant

Then Laplace transform of f(t) is given by

L{f(t)}=L(e^{at})= $\frac{1}{s-a}$ *

Similarly, L(e^{-at})= $\frac{1}{s+a}$ *

Ex: $L(e^t)=\frac{1}{s-1}$, $L(e^{-t})=\frac{1}{s+1}$, $L(e^{-2t})=\frac{1}{s+2}$, …….etc

(iv) Laplace Transform of trigonometric function:

Let f(t)=sin at

Then Laplace transform of f(t) is given by

$$L(\sin at) = \frac{a}{s^2 + a^2}, \text{where a is constant} \qquad *$$

Ex: $L(\sin t) = \frac{1}{s^2+1}, L(\sin 2t) = \frac{2}{s^2+4}, L(\sin 5t) = \frac{5}{s^2+25}$ ....etc

Let f(t)=cos at

Then Laplace transform of f(t) is given by

$$L(\cos at) = \frac{s}{s^2 + a^2}, \text{where a is constant} \qquad *$$

Ex: $L(\cos t) = \frac{s}{s^2+1}, L(\cos 2t) = \frac{s}{s^2+4}, L(\cos 3t) = \frac{s}{s^2+9}..........etc$

(v)Laplace Transform of hyperbolic function:

Let f(t)=sinh at (reads as sine hyperbolic)

Then Laplace transform of f(t) is given by

$$L(\sinh at) = \frac{a}{s^2 - a^2}, \text{where a is constant} \qquad *$$

Ex: $L(\sinh t) = \frac{1}{s^2-1}, L(\sinh 2t) = \frac{2}{s^2-4}, L(\sinh 5t) = \frac{5}{s^2-25}$ ....etc

Let f(t)=cosh at (reads as cosine hyperbolic)

Then Laplace transform of f(t) is given by

$$L(\cosh at) = \frac{s}{s^2 - a^2}, \text{where a is constant} \qquad *$$

Ex : $L(\cosh t) = \frac{s}{s^2-1}, L(\cosh 3t) = \frac{s}{s^2-9}, L(\cosh 4t) = \frac{s}{s^2-16}..........etc$

First Shifting Theorem:

$$If \quad L\{f(t)\} = \bar{f}(s), \text{then} \quad L\{e^{at} f(t)\} = \bar{f}(s-a), s-a > 0$$

Proof: From the def^n, we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \bar{f}(s) \Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-st}.e^{at} \ f(t) dt$$

$$= \int_0^\infty e^{-(s-a)t} \ f(t)dt = \bar{f}(s-a) \quad \text{(comparing def}^{\,n}\text{)}$$

Similarly

$$If \quad L\{f(t)\} = \bar{f}(s)\text{, then} \quad L\{e^{-at} f(t)\} = \bar{f}(s+a),$$

(i) $\quad L(e^{at}) = \dfrac{1}{s-a} \left(\because L(1) = \dfrac{1}{s}\right)$

(ii) $\quad L(e^{at} t^n) = \dfrac{n!}{(s-a)^{n+1}}, n = 0,1,2,.........$

(iii) $\quad L(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}$

(iv) $\quad L(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

(v) $\quad L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$

(vi) $\quad L(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$

<u>Some trigonometric formula :</u>

(i) $\quad \text{Sin}(A+B) = \text{Sin A.Cos B} + \text{Cos A.Sin B}$

(ii) $\quad \text{Sin}(A-B) = \text{Sin A.Cos B} - \text{Cos A.Sin B}$

(iii) $\quad \text{Cos}(A+B) = \text{Cos A.Cos B} - \text{SinA.Sin B}$

(iv) $\quad \text{Cos}(A-B) = \text{Cos A.Cos B} + \text{SinA.Sin B}$

(v) $\quad \text{Sin}(A+B) + \text{Sin}(A-B) = 2\text{Sin A.Cos B}$

(vi) $\quad \text{Sin}(A+B) - \text{Sin}(A-B) = 2\text{Cos A.Sin B}$

(vii) $\quad \text{Cos}(A+B) + \text{Cos}(A-B) = 2\text{Cos A.Cos B}$

$(viii)$ $\quad \text{Cos}(A-B) - \text{Cos}(A+B) = 2\text{Sin A.Sin B}$

(ix) $\quad \text{Sin 2A} = 2\text{Sin A.Cos A}$

(x) $\quad \text{Cos 2A} = \text{Cos}^2\text{A} - \text{Sin}^2\text{A} = 1 - 2\text{Sin}^2\text{A} = 2\text{Cos}^2\text{A} - 1$

(xi) $\quad \tan^{-1}x + \cot^{-1}x = \pi/2$

(xii) $\quad \text{Sin}^2\text{A} = \dfrac{1-\text{Cos 2A}}{2}, \text{Sin}^2 2\text{A} = \dfrac{1-\text{Cos 4A}}{2},.....$

$(xiii)$ $\quad \text{Cos}^2\text{A} = \dfrac{1+\text{Cos 2A}}{2}, \text{Cos}^2 2\text{A} = \dfrac{1+\text{Cos 4A}}{2},....$

(xiv) $\quad \text{Sin}^3\text{A} = \dfrac{3\text{Sin A} - \text{Sin 3A}}{4}, \text{Cos}^3\text{A} = \dfrac{3\text{Cos A} + \text{Cos 3A}}{4}$

Some Derivative formula:

$(i)\quad \frac{d}{dx}(\sin x)=\cos x, \frac{d}{dx}(\cos x)=-\sin x, \frac{d}{dx}(\sin 2x)=2\cos 2x, \frac{d}{dx}(\cos 3x)=-3\sin 3x, ...$

$(ii)\quad \frac{d}{dx}(uv)=u.\frac{dv}{dx}+v.\frac{du}{dx} \quad (product\ rule)$

Some

$(iii)\quad \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v.\frac{du}{dx}-u.\frac{dv}{dx}}{v^2} \quad (division\ rule)$

$(iv)\quad \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2},\ \frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$

Integration formula:

$(i)\quad \int \sin x\, dx = -\cos x + c\ , \int \cos x\, dx = \sin x + c$

$(ii)\quad \int \sin 2x\, dx = -\frac{\cos 2x}{2}+c\ ,\ \int \cos 3x\, dx = \frac{\sin 3x}{3}+c, ...$

$(iii)\quad \int \frac{1}{1+x^2}dx = \tan^{-1}x + c \quad , \quad \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

Algebraic Properties of Laplace transformation:

$(i)\quad L(u \pm v \pm w \pm ....) = L(u) \pm L(v) \pm ....$

$(ii)\quad L(k.f(t)) = k.L(f(t)), \text{where}\quad k \quad \text{is constant}$

Ex 1 : *Find Laplace transforms of the following functions:*

(i) $1+t^2-3t$, (ii) $(1+t)^2$, (iii) cos t -3sin t, (iv) $\sin^2 t$, (v) $\cos^2 t$
(v) $\text{Sin}^2\ 2t$, (vii) $\cos^2 3t$,(viii) $(\cos t - \sin t)^2$, (ix) $1+e^{-t}$, (x) sin 3t. cos 2t
(xi) cos 4t.cos t ,(xii) sin 3t.sin t, (xiii) cos(at+b), (xiv) cosh 2t- sinh t

(xv) $\text{Sin}^3 t$, (xvi) $\text{Cos}^3 t$, (xvii) $\text{Sin}^3 2t$, (xviii) $\text{Cos}^3 3t$.

Sol^n :

(i) $\quad \mathrm{L}(1+t^2-3t)=\mathrm{L}(1)+\mathrm{L}(t^2)-\mathrm{L}(3t)=\frac{1}{s}+\frac{2!}{s^{2+1}}-3.\frac{1}{s^2}=\frac{1}{s}+\frac{2}{s^3}-\frac{3}{s^2}$

(ii) $\quad \mathrm{L}(1+t)^2=L(1+2t+t^2)=L(1)+L(2t)+L(t^2)=\frac{1}{s}+\frac{2}{s^2}+\frac{2}{s^3}$

(iii) $\quad L(\cos t-3\sin t)=L(\cos t)-L(3\sin t)=\frac{s}{s^2+1}-\frac{3}{s^2+1}$

(iv) $\quad L(\sin^2 t)=L\left(\frac{1-\cos 2t}{2}\right)=\frac{1}{2}\{L(1)-L(\cos 2t)\}=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$

(v) $\quad L(\cos^2 t)=L\left(\frac{1+\cos 2t}{2}\right)=\frac{1}{2}\{L(1)+L(\cos 2t)\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$

(vi) $\quad L(\sin^2 2t)=L\left(\frac{1-\cos 4t}{2}\right)=\frac{1}{2}\{L(1)-L(\cos 4t)\}=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+16}\right)$

(vii) $\quad L(\cos^2 3t)=L\left(\frac{1+\cos 6t}{2}\right)=\frac{1}{2}\{L(1)+L(\cos 6t)\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+36}\right)$

$(viii)$ $\quad L(\cos t+\sin t)^2=L(\cos^2 t+\sin^2 t+2\sin t.\cos t)$

$$=L(1+\sin 2t)=L(1)+L(\sin 2t)=\frac{1}{s}+\frac{2}{s^2+4}$$

(ix) $\quad L(1+e^{-t})=L(1)+L(e^{-t})=\frac{1}{s}+\frac{1}{s+1}$

(x) $\quad L(\sin 3t.\cos 2t)=\frac{1}{2}L(2\sin 3t.\cos 2t)$

$$=\frac{1}{2}L\{\sin(3t+2t)+\sin(3t-2t)\}$$

$$=\frac{1}{2}L(\sin 5t+\sin t)=\frac{1}{2}\left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)$$

(xi) $\quad L(\cos 4t.\cos t)=\frac{1}{2}L(2\cos 4t.\cos t)$

$$=\frac{1}{2}L(\cos(4t+t)+\cos(4t-t))$$

$$=\frac{1}{2}L(\cos 5t+\cos 3t)=\frac{1}{2}\left(\frac{s}{s^2+25}+\frac{s}{s^2+9}\right)$$

$(xii)\quad L(\sin 3t.\sin t)=\frac{1}{2}L(2\sin 3t.\sin t)$

$$=\frac{1}{2}L\{\cos(3t-t)-\cos(3t+t)\}$$

$$=\frac{1}{2}L(\cos 2t-\cos 4t)=\frac{1}{2}\left(\frac{s}{s^2+4}-\frac{s}{s^2+16}\right)$$

$(xiii)\quad L\{\cos(at+b)\}=L(\cos at.\cos b-\sin at.\sin b)$

$$=L(\cos at.\cos b)-L(\sin at.\sin b)$$

$$=\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}$$

$(xiv)\quad L\{\cosh 2t-\sinh t\}=L(\cosh 2t)-L(\sinh t)=\frac{s}{s^2-4}-\frac{1}{s^2-1}$

$(xv)\quad L(\sin^3 t)=L\left(\frac{3\sin t-\sin 3t}{4}\right)=\frac{1}{4}\left(\frac{3}{s^2+1}-\frac{3}{s^2+9}\right)$

$(xvi)\quad L(\cos^3 t)=L\left(\frac{3\cos t+\cos 3t}{4}\right)=\frac{1}{4}\left(\frac{3s}{s^2+1}+\frac{s}{s^2+9}\right)$

$(xvii)\quad L(\sin^3 2t)=L\left(\frac{3\sin 2t-\sin 6t}{4}\right)=\frac{1}{4}\left(3.\frac{2}{s^2+4}-\frac{6}{s^2+36}\right)$

$$=\frac{1}{4}\left(\frac{6}{s^2+4}-\frac{6}{s^2+36}\right)$$

$(xviii)\quad L(\cos^3 3t)=L\left(\frac{3\cos 3t+\cos 9t}{4}\right)=\frac{1}{4}\left(\frac{3s}{s^2+9}+\frac{s}{s^2+81}\right)$

Exercises-1

Find Laplace transform of the following functions:

(i) 2-3t+e^{2t}
(ii) $(1-t^2)^2$ (iii) $(e^t+1)^2$
(iv)$(\sin t-\cos t)^2$ (v)$\text{Sin}^2 3t$ (vi)$\text{Cos}^2 2t$
(vii)Sin4t.cos2t (viii)Cos3t.cos2t
(iX)Sin3t.sin2t
(x)Sin(at+b)
(xi)$\text{Sin}^3 3t$

Ex 2: Find Laplace transform of the following functions

(i) te^{-t} (ii) t^2e^{2t} (iii) $e^{2t}t^5$ (iv) $e^{-t}\sin t$ (v) $e^{2t}\cos 2t$ (vi) $e^{3t}\sin^2 t$
(vi) $e^{-2t}\cos^2 t$ (viii) $e^{-4t}\sin 2t \cos t$ (ix) $e^{t}\cos 3t \cos t$ (x) $e^{-t}(\cosh 2t - \sinh t)$
(xi) $(1+t)e^{-t}$

Sol^n:

(i) We have $L(t) = \frac{1}{s^2}$

Using first shifting rule, $L(te^{-t}) = \frac{1}{(s+1)^2}$ $[i.e.\ s \to s-(-1) = s+1]$

(ii) We have $L(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

Using first shifting rule, $L(t^2e^{2t}) = \frac{2}{(s-2)^3}$

(iii) We have $L(t^5) = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$

Using first shifting rule, $L(e^{2t}t^5) = \frac{120}{(s-2)^6}$

(iv) We have $L(\sin t) = \frac{1}{s^2+1}$

Using first shifting rule, $L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$

(v) We have $L(\cos 2t) = \frac{s}{s^2+4}$

Using first shifting rule, $L(e^{2t}\cos 2t) = \frac{s-2}{(s-2)^2+4} = \frac{s-2}{s^2-4s+8}$

(vi) We have $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

Using first shifting rule, $L(e^{3t}\sin^2 t) = \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{(s-3)^2+4}\right)$

(vii) We have $L(\cos^2 t) = L\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$

Using first shifting rule, $L(e^{-2t}\cos^2 t) = \frac{1}{2}\left(\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right)$

$(viii)$ We have $L(\sin 2t\cos t)=\frac{1}{2}L(2\sin 2t\cos t)$

$$=\frac{1}{2}L(\sin 3t+\sin t)=\frac{1}{2}\left(\frac{3}{s^2+9}+\frac{1}{s^2+1}\right)$$

Using first shifting rule, $L(e^{-4t}\sin 2t\cos t)=\frac{1}{2}\left(\frac{3}{(s+4)^2+9}+\frac{1}{(s+4)^2+1}\right)$

(ix) We have $L(\cos 3t\cos t)=\frac{1}{2}L(2\cos 3t\cos t)$

$$=\frac{1}{2}L(\cos 4t+\cos 2t)=\frac{1}{2}\left(\frac{s}{s^2+16}+\frac{s}{s^2+4}\right)$$

Using first shifting rule, $L(e^{t}\cos 3t\cos t)=\frac{1}{2}\left(\frac{s-1}{(s-1)^2+16}+\frac{s-1}{(s-1)^2+4}\right)$

(x) We have $L(\cosh 2t - \sinh t)=\left(\frac{s}{s^2-4}-\frac{1}{s^2-1}\right)$

Using first shifting rule, $L\{e^{-t}(\cosh 2t - \sinh t)\}=\frac{s+1}{(s+1)^2-4}-\frac{1}{(s+1)^2-1}$

(xi) We have $L(1+t)=\frac{1}{s}+\frac{1}{s^2}$

Using first shifting rule, $L\{(1+t)e^{-t}\}=\frac{1}{s+1}+\frac{1}{(s+1)^2}$

Exercises-2

Find Laplace transform of the following functions:

(i) te^{2t} (ii) t^3e^{-t} (iii) $e^{-2t}\sin 3t$ (iv) $e^{t}\cos 4t$ (v) $e^{2t}\cos^2 2t$ (vi) $e^{-t}\sin^2 3t$ (vii) $e^{-t}\sin^3 t$ (viii) $e^{t}\cos^3 t$ (ix) $e^{-t}(1-t^2)$ (x) $e^{2t}\sinh t$

(xi) $e^{t}(\cosh 2t - \sin t)$(xii) $e^{2t}(3t^5-\cos 4t)$(xiii) $e^{-t}\sin 4t\cos 2t$(xiv) $e^{-3t}\sin 5t\sin 3t$

Change of scale property:

If $L\{f(t)\}=\bar{f}(s)$, then

$$L\{f(at)\}=\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Laplace Transforms of Derivatives:

If $L\{f(t)\} = \bar{f}(s)$ and $f'(t)$ is contineous , then

$L\{f'(t)\} = s\bar{f}(s) - f(0)$ *

$L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$ *

$L\{f'''(t)\} = s^3\bar{f}(s) - s^2 f(0) - sf'(0) - f''(0)$ *

and so on

Laplace Transforms of Integrals:

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$ *

Laplace Transforms of t^n f(t) :(Multiplication by t , t^2,.....)

If $L\{f(t)\} = \bar{f}(s)$, then for a positive integer n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left\{\bar{f}(s)\right\}$$

Particular ly , for n = 1, $L\{t\, f(t)\} = (-1)^1 \frac{d^1}{ds^1}\left\{\bar{f}(s)\right\} = -\frac{d}{ds}\left\{\bar{f}(s)\right\}$ *

for n = 2, $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\left\{\bar{f}(s)\right\} = \frac{d^2}{ds^2}\left\{\bar{f}(s)\right\}$ *

Laplace transform of $\frac{f(t)}{t}$ (division by t)

If $L\{f(t)\} = \bar{f}(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)ds$ *

Ex 3 : Find the Laplace transforms of the following functions : (i)tsint (ii) t cost (iii) t^2sint (iv) t^2 cost

(v) t $\sin^2$t (vi) t cos2t (vii) t e^{-t}sin4t (viii) t sin3t.cos2t (ix) t e^{-t} cosht (x) t^2 sinat

(i) Here $f(t) = \sin t$

$\therefore L(f(t)) = L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s)$

So, by using result $L(t\, f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{0-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2} \quad (by\ \ division\ \ rule)$$

(ii) Here $f(t)=\cos t$

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cos t) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left(\frac{1.(s^2+1)-s.2s}{(s^2+1)^2}\right) = \frac{s^2-1}{(s^2+1)^2}$$

(iii) Here $f(t)=\sin t$

$$\therefore L(f(t)) = L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t^2 f(t)) = \frac{d^2}{ds^2}\left(\bar{f}(s)\right)$, we get

$$L(t^2 \sin t) = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right) = \frac{2.(s^2+1)^2 - 2s.2(s^2+1).2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)\{2(s^2+1)-8s^2\}}{(s^2+1)^4} = \frac{2-6s^2}{(s^2+1)^3}$$

(iv) Here $f(t)=\cos t$

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t^2 f(t)) = \frac{d^2}{ds^2}\left(\bar{f}(s)\right)$, we get

$$L(t^2 \cos t) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{1.s^2+1-s.2s}{(s^2+1)^2}\right) = \frac{d}{ds}\frac{1-s^2}{(s^2+1)^2}$$

$$= \frac{-2s.(s^2+1)^2-(1-s^2).2(s^2+1).2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)(-2s^3-2s-4s+4s^3)}{(s^2+1)^4} = \frac{2s^3-6s}{(s^2+1)^3}$$

(v) Here $f(t) = \sin^2 t$

$$\therefore L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \bar{f}(s)$$

So, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\sin^2 t) = -\frac{d}{ds}\left\{\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{s^2} - \frac{1.s^2+4-s.2s}{(s^2+4)^2}\right) = \frac{1}{2s^2} + \frac{4-s^2}{2(s^2+4)^2}$$

(vi) Here $f(t) = \cos 2t$

$$\therefore L(\cos 2t) = \frac{s}{s^2+4} = \bar{f}(s)$$

So, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cos 2t) = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{1.s^2+4-s.2s}{(s^2+4)^2} = \frac{4-s^2}{(s^2+4)^2}$$

(vii) We have $L(\sin 4t) = \frac{4}{s^2+16}$

So, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = -\frac{0-4.2s}{(s^2+16)^2} = \frac{8s}{(s^2+16)^2}$$

Now by using first shifting property, we get

$$L(e^{-t}\,t\sin 4t) = \frac{8(s+1)}{\left[(s+1)^2+16\right]^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$$

(ix) We have $L(\cosh t) = \frac{s}{s^2-1}$

So, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cosh t) = -\frac{d}{ds}\left(\frac{s}{s^2-1}\right) = -\frac{1.s^2-1-s.2s}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

Now by using first shifting property, we get

$$L(e^{-t}\,t\cosh t) = \frac{(s+1)^2+1}{\left[(s+1)^2-1\right]^2} = \frac{s^2+2s+2}{(s^2+2s)^2}$$

$(viii)$ Here $\mathrm{f}(\mathrm{t}) = \sin 3t.\cos 2t$

$$\therefore \mathrm{L}(\sin 3t.\cos 2t) = \frac{1}{2}\mathrm{L}(2\sin 3t.\cos 2t)$$

$$= \frac{1}{2}\mathrm{L}(\sin 5\mathrm{t} + \sin \mathrm{t}) = \frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right) = \bar{\mathrm{f}}(s)$$

So, using result $\mathrm{L}(\mathrm{t}\,\mathrm{f}(\mathrm{t})) = -\frac{d}{ds}\left(\bar{\mathrm{f}}(s)\right)$, we get

$$\mathrm{L}(\mathrm{t}.\sin 3\mathrm{t}\cos 2\mathrm{t}) = -\frac{d}{ds}\frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right) = -\frac{1}{2}\left(\frac{0-5.2s}{(s^2+25)^2} + \frac{0-2s}{(s^2+1)^2}\right)$$

$$= -\frac{1}{2}\left(\frac{-10s}{(s^2+25)^2} + \frac{-2s}{(s^2+1)^2}\right) = \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

(x) Here $\mathrm{f}(\mathrm{t}) = \sin \mathrm{at}$

$$\therefore \mathrm{L}(\sin \mathrm{at}) = \frac{a}{s^2+a^2} = \bar{\mathrm{f}}(s)$$

So by using result $\mathrm{L}(\mathrm{t}^2\,\mathrm{f}(\mathrm{t})) = \frac{d^2}{ds^2}\left(\bar{\mathrm{f}}(s)\right)$, we get

$$\mathrm{L}(\mathrm{t}^2 \sin at) = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{0-a.2s}{(s^2+a^2)^2}\right) = -2a\frac{d}{ds}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$= -2a\left(\frac{1.(s^2+a^2)^2 - s.2(s^2+a^2).2s}{(s^2+a^2)^4}\right)$$

$$= \frac{-2a(s^2+a^2)\{s^2+a^2-4s^2\}}{(s^2+a^2)^4} = \frac{-2a(a^2-3s^2)}{(s^2+a^2)^3}$$

Exercises-3

Find Laplace transforms of the following functions: (i)t sin2t (ii) t cos 3t (iii) t^2 cos2t (iv)t e^t sin2t

(v) t e^{-2t} cos t (vi) te^{-t} sinh 2t (vii)t cos4t cos t (viii)t^2 e^{--t} cos t

Ex 4 : Find the Laplace transforms of the following functions.

$(i)\ \frac{1-e^t}{t}$, $(ii)\ \frac{e^{-t}-1}{t}$, $(iii)\ \frac{e^t-e^{-t}}{t}$, $(iv)\ \frac{e^{-at}-e^{-bt}}{t}$

$(v)\ \frac{1-\cos 2t}{t}$, $(vi)\ \frac{\cos 2t-\cos 3t}{t}$, $(vii)\ \frac{\sin t}{t}$ $(viii)\ \frac{\sin^2 t}{t}$

$(ix)\ \frac{e^{-t}\sin t}{t}$, $(x)\ \frac{\sin at}{t}$, $(xi)\ \frac{e^{at}-\cos bt}{t}$

Sol^n:

(i) Here $f(t)=1-e^t$

$$\bar{f}(s)=L(1-e^t)=\frac{1}{s}-\frac{1}{s-1}$$

$$\therefore L\left(\frac{1-e^t}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$$

$$=[\log s-\log(s-1)]_s^\infty=\left[\log\frac{s}{s-1}\right]_s^\infty=\log\frac{s-1}{s}$$

(ii) Here $f(t)=e^{-t}-1$

$$\bar{f}(s)=L(e^{-t}-1)=\frac{1}{s+1}-\frac{1}{s}$$

$$\therefore L\left(\frac{e^{-t}-1}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty\left(\frac{1}{s+1}-\frac{1}{s}\right)ds$$

$$=[\log(s+1)-\log s]_s^\infty=\left[\log\frac{s+1}{s}\right]_s^\infty=\log\frac{s}{s+1}$$

(iii) Here $f(t)=e^t-e^{-t}$

$$\bar{f}(s)=L(e^t-e^{-t})=\frac{1}{s-1}-\frac{1}{s+1}$$

$$\therefore L\left(\frac{e^t-e^{-t}}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds$$

$$=[\log(s-1)-\log(s+1)]_s^\infty=\left[\log\frac{s-1}{s+1}\right]_s^\infty=\log\frac{s+1}{s-1}$$

(iv) Here $f(t)=e^{-at}-e^{-bt}$

$$\bar{f}(s)=L(e^{-at}-e^{-bt})=\frac{1}{s+a}-\frac{1}{s+b}$$

$$\therefore L\left(\frac{e^{-at}-e^{-bt}}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=[\log(s+a)-\log(s+b)]_s^\infty=\left[\log\frac{s+a}{s+b}\right]_s^\infty=\log\frac{s+b}{s+a}$$

(v) Here $f(t) = 1 - \cos 2t$

$$\bar{f}(s) = L(1 - \cos 2t) = \frac{1}{s} - \frac{s}{s^2 + 4}$$

$$\therefore L\left(\frac{1 - \cos 2t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2 + 4\right)\right]_s^\infty = \left[\log s - \log\sqrt{s^2 + 4}\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2 + 4}}\right]_s^\infty = \log\frac{\sqrt{s^2 + 4}}{s}$$

(vi) Here $f(t) = \cos 2t - \cos 3t$

$$\bar{f}(s) = L(\cos 2t - \cos 3t) = \frac{s}{s^2 + 4} - \frac{s}{s^2 + 9}$$

$$\therefore L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9}\right) ds$$

$$= \left[\frac{1}{2}\log\left(s^2 + 4\right) - \frac{1}{2}\log\left(s^2 + 9\right)\right]_s^\infty$$

$$= \frac{1}{2}\log\left[\frac{s^2 + 4}{s^2 + 9}\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2 + 9}{s^2 + 4}\right)$$

(vii) Here $f(t) = \sin t$

$$\bar{f}(s) = L(\sin t) = \frac{1}{s^2 + 1}$$

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2 + 1}\,ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$(viii)$ Here $f(t) = \sin^2 t$, $\bar{f}(s) = L(\sin^2 t) = L\left(\frac{1 - \cos 2t}{2}\right)$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4}\right)$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4}\right) ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2 + 4\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2 + 4}\right]_s^\infty = \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2 + 4}}\right]_s^\infty = \frac{1}{2}\log\left(\frac{\sqrt{s^2 + 4}}{s}\right)$$

(ix) Here $f(t) = e^{-t}\sin t$

$$\bar{f}(s) = L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1} \quad (\text{by shifting rule})$$

$$\therefore L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}(s+1)$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(x) Here $f(t) = \sin at$

$$\bar{f}(s) = L(\sin at) = \frac{a}{s^2+a^2}$$

$$\therefore L\left(\frac{\sin at}{t}\right) = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds = a.\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

(xi) Here $f(t) = e^{at} - \cos bt$

$$\bar{f}(s) = L(e^{at} - \cos bt) = \frac{1}{s-a} - \frac{s}{s^2+b^2}$$

$$\therefore L\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty = \left[\log(s-a) - \log\sqrt{s^2+b^2}\right]_s^\infty$$

$$= \left[\log\frac{s-a}{\sqrt{s^2+b^2}}\right]_s^\infty = \log\frac{\sqrt{s^2+b^2}}{s-a}$$

Exercises-4

Find Laplace transforms of the following functions:

$(i)\ \frac{1-e^{at}}{t}$ $\quad (ii)\ \frac{e^{at}-e^{bt}}{t}$ $\quad (iii)\ \frac{1-\cos t}{t}$ $\quad (iv)\ \frac{\cos at - \cos bt}{t}$

$(v)\ \frac{\sin 2t}{t}$ $\quad (vi)\ \frac{e^{t}\sin t}{t}$ $\quad (vii)\ \frac{e^{-t}-e^{-3t}}{t}$

Ex 5 : Find the Laplace transforms of the following functions :

$(i)\int_0^t e^{-t}\cos t\,dt \qquad (ii)\int_0^t \frac{\sin t}{t}\,dt \quad (iii)\int_0^t e^t\left(\frac{\sin t}{t}\right)dt$

$(iv)\int_0^t \frac{\cos at-\cos bt}{t}\,dt$

Sol^n^:

(i) Here $\mathrm{f}(\mathrm{t}) = e^{-t}\cos t$

$$\mathrm{L}\{\mathrm{f}(\mathrm{t})\} = \mathrm{L}\left(e^{-t}\cos t\right) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2} = \bar{\mathrm{f}}(s)$$

$$\therefore \mathrm{L}\left(\int_0^t e^{-t}\cos t\,dt\right) = \frac{\bar{\mathrm{f}}(s)}{s} = \frac{1}{s}\cdot\frac{s+1}{s^2+2s+2} = \frac{s+1}{s(s^2+2s+2)}$$

(ii) Here $\mathrm{f}(\mathrm{t}) = \frac{\sin t}{t}$

$$\mathrm{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{\mathrm{f}}(s)ds = \int_s^\infty \mathrm{L}(\sin \mathrm{t})ds \qquad (by\ division\ rule)$$

$$= \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = \bar{\mathrm{f}}(s)$$

$$\therefore \mathrm{L}\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{\bar{\mathrm{f}}(s)}{s} = \frac{\cot^{-1}s}{s}$$

(iii) Here $\mathrm{f}(\mathrm{t}) = \left(e^t\left(\frac{\sin t}{t}\right)\right)$

$$\text{Now } \mathrm{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{\mathrm{f}}(s)ds = \int_s^\infty \mathrm{L}(\sin \mathrm{t})ds \qquad (by\ division\ rule)$$

$$= \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\text{So, } \mathrm{L}\left(e^t\left(\frac{\sin t}{t}\right)\right) = \cot^{-1}(s-1) = \bar{\mathrm{f}}(s) \quad (by\ \ first\ \ shifting\ \ rule)$$

$$\therefore \mathrm{L}\left(\int_0^t e^t\left(\frac{\sin t}{t}\right)dt\right) = \frac{\bar{\mathrm{f}}(s)}{s} = \frac{\cot^{-1}(s-1)}{s}$$

(iv) Here $f(t)=\dfrac{\cos at - \cos bt}{t}$

Now $L\left(\dfrac{\cos at - \cos bt}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty L(\cos at - \cos bt)ds$

$$=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$=\left[\frac{1}{2}\log\left(s^2+a^2\right)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)=\bar{f}(s)$$

$$\therefore L\left(\int_0^t\frac{\cos at - \cos bt}{t}dt\right)=\frac{\bar{f}(s)}{s}=\frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Exercise-5

Find Laplace transforms of the following functions:

$(i)\ \int_0^t e^{2t}\sin t\ dt$ $(ii)\int_0^t \sin\ 2t\cos t\ dt$ $(iii)\int_0^t t\sin\ 2t\ dt$

$(iv)\ \int_0^t t\,e^{-t}\sin\ 4t\,dt$ $(v)\int_0^t\dfrac{\cos 2t - \cos 3t}{t}dt$

(vi) By using the Laplace transform of sin at, find Laplace transform of cos at.

Ex 6: By using the Laplace transform of cos at, find Laplace transform of sin at.

Sol^n : Let $f(t)=\cos at$. Then $L(f(t))=\dfrac{s}{s^2+a^2}=\bar{f}(s)$

Now $f'(t)=-a\sin at$

$$\Rightarrow L\left(f'(t)\right)=-a\,L(\sin at)$$

$$\Rightarrow L(\sin at)=-\frac{1}{a}L\left(f'(t)\right)$$

$$=-\frac{1}{a}\left(s\bar{f}(s)-f(0)\right)\quad(by\ Laplace\ transform\ of\ derivative)$$

$$=-\frac{1}{a}\left(s.\frac{s}{s^2+a^2}-1\right)=-\frac{1}{a}\left(\frac{s^2-s^2-a^2}{s^2+a^2}\right)$$

$$=\frac{a}{s^2+a^2}$$

Evaluation of integrals using def^n of Laplace Transform :

We know from definition of Laplace Transform that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \text{ where } s \text{ is real or complex}$$

Now we shall evaluate the integrals using above result.

Ex 6: Evaluate following integrals:

$(i) \int_0^\infty e^{-t}\, t\, dt$

$$\text{Sol}^n : \int_0^\infty e^{-t}\, t\, dt = L(t) \text{ where } s = 1 \text{, by definition}$$

$$= \frac{1}{s^2} = \frac{1}{1} = 1$$

$(ii) \int_0^\infty t\, e^{-2t} \sin t\, dt$

$$\text{Sol}^n : \int_0^\infty te^{-2t} \sin t\, dt = L(t\sin t) \text{ where } s = 2 \text{, by definition}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

$$(iii) \int_0^\infty t\, e^{3t} \cos t\, dt = L(t\cos t) \text{ where } s = -3\text{, by definition}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left(\frac{1.s^2+1-s.2s}{(s^2+1)^2}\right)$$

$$= \frac{s^2-1}{(s^2+1)^2} = \frac{9-1}{(9+1)^2} = \frac{8}{100} = \frac{2}{25}$$

Exercise - 6

Evaluate the following integrals :

$(i) \int_0^\infty t^2 e^t dt$ $\quad (ii) \int_0^\infty t\, e^{-t} \sin 2t\, dt$ $\quad (iii) \int_0^\infty e^{-2t}\, t \cos t\, dt$

$(iv) \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$ $\quad (v) \int_0^\infty \frac{e^{-t}\sin^2 t}{t} dt$

Inverse Laplace Transforms:

As before , if $L\{f(t)\} = \bar{f}(s)$, then $L^{-1}\left\{\bar{f}(s)\right\} = f(t)$

Here , the symbol L^{-1} stands for inverse laplace

Formula :

$(i)\ L^{-1}\left(\frac{1}{s}\right) = 1$

$(ii)\ L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$(iii)\ L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!},\ n = 1,2,3,.......$

$(iv)\ L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{(n-1)!},\ n = 1,2,3,........$

$(v)\ L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

$(vi)\ L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$(vii)\ L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$(viii)\ L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$

$(ix)\ L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

$(x)\ L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

$(xi)\ L^{-1}\left(\frac{a}{(s-b)^2+a^2}\right) = e^{bt}\sin at$

$(xii)\ L^{-1}\left(\frac{1}{(s-b)^2+a^2}\right) = \frac{1}{a}e^{bt}\sin at$

$(xiii)\ L^{-1}\left(\frac{s-b}{(s-b)^2+a^2}\right) = e^{bt}\cos at$

$(xiv)\ L^{-1}\left(\frac{s}{(s^2+b^2)^2}\right) = \frac{1}{2b}t\sin bt$

Ex -1: Find inverse Laplace transform of the following :

$$(i)\frac{1}{s^4},\ (ii)\frac{2}{s-1},\ (iii)\frac{3}{s+5},\ (iv)\frac{2s}{s^2+4},\ (v)\frac{s-1}{s^2+9}$$

Sol^n : $(i)L^{-1}\left(\frac{1}{s^4}\right)=\frac{t^{4-1}}{(4-1)!}=\frac{t^3}{3!}=\frac{t^3}{6}$

$$(ii)L^{-1}\left(\frac{2}{s-1}\right)=2L^{-1}\left(\frac{1}{s-1}\right)=2e^t$$

$$(iii)L^{-1}\left(\frac{3}{s+5}\right)=3L^{-1}\left(\frac{1}{s+5}\right)=3e^{-5t}$$

$$(iv)L^{-1}\left(\frac{2s}{s^2+4}\right)=2L^{-1}\left(\frac{s}{s^2+4}\right)=2\cos 2t$$

$$(v)L^{-1}\left(\frac{s-1}{s^2+9}\right)=L^{-1}\left(\frac{s}{s^2+9}\right)-L^{-1}\left(\frac{1}{s^2+9}\right)=\cos 3t-\frac{1}{3}\sin 3t$$

Ex - 2: Find inverse Laplace transform of the following :

$$(i)\frac{1}{s^2-2s+5},\ (ii)\frac{s}{s^2+2s+10},\ (iii)\frac{3s+7}{s^2-2s-3}$$

Sol^n : $(i)L^{-1}\left(\frac{1}{s^2-2s+5}\right)=L^{-1}\left(\frac{1}{s^2-2s+1-1+5}\right)$

$$=L^{-1}\left(\frac{1}{(s-1)^2+4}\right)=\frac{1}{2}e^t\sin 2t$$

$$(ii)L^{-1}\left(\frac{s}{s^2+2s+10}\right)=L^{-1}\left(\frac{s}{s^2+2s+1-1+10}\right)$$

$$=L^{-1}\left(\frac{s}{(s+1)^2+9}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2+9}\right)$$

$$=L^{-1}\left(\frac{s+1}{(s+1)^2+9}\right)-L^{-1}\left(\frac{1}{(s+1)^2+9}\right)$$

$$=e^{-t}\cos 3t-\frac{1}{3}e^{-t}\sin 3t$$

$$(iii)L^{-1}\left(\frac{3s+7}{s^2-2s-3}\right)=L^{-1}\left(\frac{3s}{s^2-2s-3}\right)+L^{-1}\left(\frac{7}{s^2-2s-3}\right)$$

$$=L^{-1}\left(\frac{3s}{(s-1)^2-4}\right)+L^{-1}\left(\frac{7}{(s-1)^2-4}\right)=L^{-1}\left(\frac{3(s-1+1)}{(s-1)^2-4}\right)+L^{-1}\left(\frac{7}{(s-1)^2-4}\right)$$

$$=L^{-1}\left(\frac{3(s-1)}{(s-1)^2-4}\right)+L^{-1}\left(\frac{10}{(s-1)^2-4}\right)$$

$$=3e^t\cosh 2t+10.\frac{1}{2}e^t\sinh 2t$$

$$=3e^t\cosh 2t+5e^t\sinh 2t$$

Reference Books:

Higher engineering mathematics By Dr B.S. Grewal (khanna publishers)