KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -III

PART-1

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CONTENTS

CHAPTER -1
NUMERICAL METHODS

Numerical methods

Algebraic Equation

An equation in the form of $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0$ *is called an* algebraic *equation of* degree *n*.

$Ex: 2x^3 + 3x^2 + x - 2 = 0$

Transcendental Equation

If an equation involves trigonometry, logarithm *and* exponential *function then it is called transcendental equation.*

Ex :

$\cos x - xe^x = 0$

$\log_e x - e^x + 5 = 0$

Roots of an equation

If $x = a$ *satisfies to an equation* $f(x) = 0$ *then* $x = a$ *is called the root of an equation.*

Ex : *solve* $x^2 - 4 = 0$

Ans : $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Hence $x = 2$ *is one of the root of the given equation* $x^2 - 4 = 0$

Method of solution two types

(1) *Direct method*

(2) *Itarative method*

Direct method

In Direct metod we will get exact value of the roots of an equation in finite number of steps.

Ex : $a_0x + a_1 = 0$

$\Rightarrow a_0x = -a_1 \Rightarrow x = \dfrac{-a_1}{a_0}$

Ex : $ax^2 + bx + c = 0, a \neq 0$

so $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Itarative method

In Iterative method we will get the better approximate root of an equation considering to an initial approximation.

Here the roots can be obtained by using three methods

(1) *Bisection method*

(2) *Secant method* (Regula falsi method)

(3) *Newton Raphson method*

Intermediate value property

If $f(x)$ *is a continuous function in the* interval $[a,b]$ *and* $f(a), f(b)$ *have different signs i.e* $f(a)f(b)<0(-ve)$ *then the equation* $f(x)=0$ *has at least one root between* $x=a$ *and* $x=b$.

Bisection method

Let the given equation is $f(x)=0$. *Find the value of* $f(x)$ *in different* integral *values of* $x=a$ *and* $x=b$, *i.e* $f(a)$ *and* $f(b)$. *If* $f(a)$ *and* $f(b)$ *have opposite signs i.e* $f(a)f(b)<0(-ve)$ *then the roots of the equation* $f(x)=0$ *lies between* a *and* b.

In bisection *method the other roots can be obtained as*

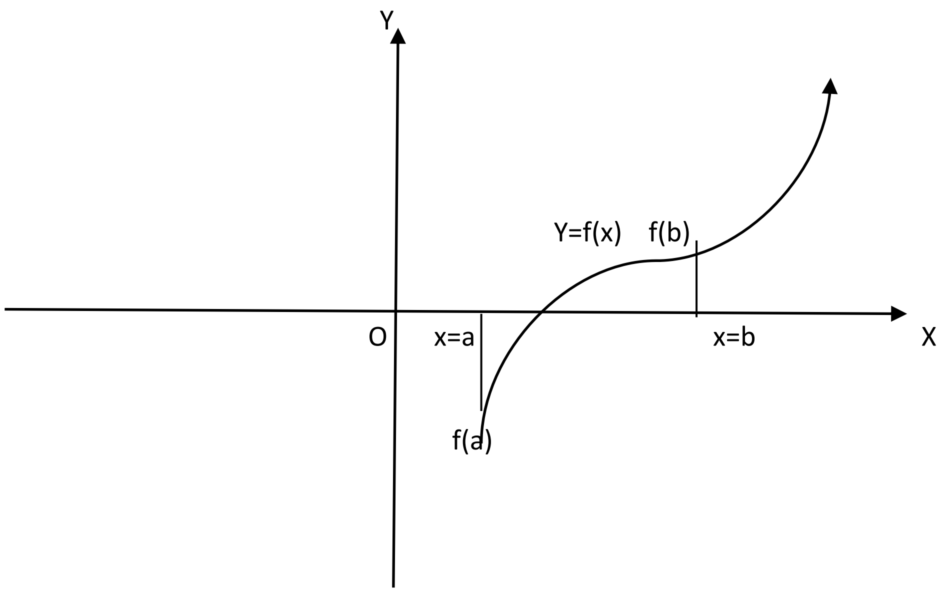
$$x_1=\frac{a+b}{2}$$

find $f(x_1)$ *if* $f(a)f(x_1)<0$ *then the roots lies between* a *and* x_1

so $x_2=\frac{a+x_1}{2}$ *otherwise the roots lies between* x_1 *and* b *so* $x_2=\frac{x_1+b}{2}$

continue the process till we get the correct root of the equation.

GRAPH: Here $f(a)f(b)<0$



Ex : *Find the* 1*st approximation root of the equation* $x^3 - 5x + 1 = 0$ using the bisection *method*.

$Ans : Let\ f(x) = x^3 - 5x + 1$

$f(0) = 0^3 - 5 \times 0 + 1 = 1(+ve)$

$f(1) = 1^3 - 5 \times 1 + 1 = -3(-ve)$

$Now\ f(0)\ f(1) < 0$, *so the roots lies between* 0 *and* 1

$let\ x_1 = \dfrac{0+1}{2} = 0.5$

Ex : *Find the first approximation root of an equation* $x^3 - 4x - 9 = 0$ using *bi* sec*tion method*.

Ex : *Find the first approximation root of an equation* $x^3 - 3x - 5 = 0$ using bisection *method*.

Ex : *Find the root of the equation* $x^3 - 4x - 9 = 0$ using bisection *method in four stages.*

$Ans : Given\ x^3 - 4x - 9 = 0$

$Let\ f(x) = x^3 - 4x - 9$

$f(0) = 0 - 0 - 9 = -9(-ve)$

$f(1) = 1 - 4 - 9 = -12(-ve)$

$f(2) = 8 - 8 - 9 = -9(-ve)$

$f(3) = 27 - 12 - 9 = 6(+ve)$

$f(2)\ f(3) < 0(-ve)$

So the roots lies between 2 *and* 3

$Let\ x_1 = \dfrac{2+3}{2} = 2.5$

$f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375(-ve)$

$f(2.5)\ f(3) < 0$

The roots lies between $(2.5, 3)$

$x_2 = \dfrac{2.5+3}{2} = 2.75$

$f(2.75) = (2.75)^3 - 4(2.75) - 9 = .797(+ve)$

$f(2.5)\ f(2.75) < 0$

So the roots lies between $(2.5, 2.75)$

$x_3 = \frac{2.5+2.75}{2} = 2.625$

$f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.412(-ve)$

$f(2.625)\, f(2.75) < 0$

So the roots lies between $(2.625, 2.75)$

$x_4 = \frac{2.625+2.75}{2} = 2.688$

Ex : Find the root of an equation $x^3 - x - 11 = 0$ *by* using bisection *method and the roots lies between* 2 *and* 3, *upto four steps*.

Ex : Find the root of an equation $x^4 - x - 10 = 0$ *by* using bisection *method upto five steps*.

Ex : Find the root of an equation $x^3 - 3x + 4 = 0$ *by* using bisection *method correct to two decimal places.*

Ans : Given $x^3 - 3x + 4 = 0$

$f(x) = x^3 - 3x + 4$

$f(0) = 4(+ve)$

$f(1) = 1 - 3 + 4 = 2(+ve)$

$f(2) = 8 - 6 + 4 = 6(+ve)$

$f(-1) = 6(+ve)$

$f(-2) = 2(+ve)$

$f(-3) = -14(-ve)$

$f(-2)\, f(-3) < 0$

So the roots lies between -3 *and* -2

Newton- Raphson Method : Let the given equation f(x)=0 , find the value of f(x) in different integral values of x=a and x=b , i.e. f(a) and f(b), if f(a) and f(b) have opposite signs , i.e. $f(a)f(b) < 0(-ve)$ then the roots of the equation f(x)=0 lies between a and b .

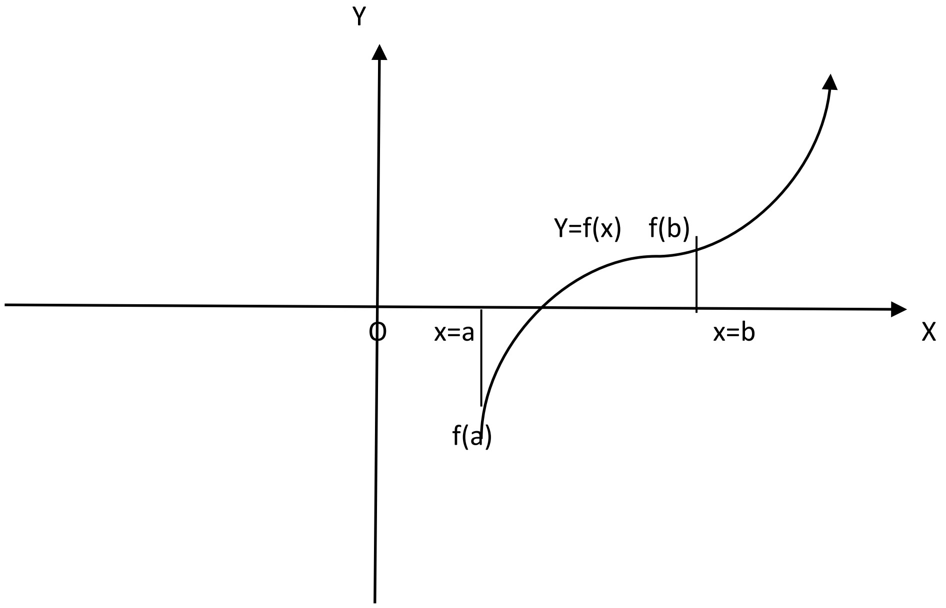
Let the initial root is $x_0 = \frac{a+b}{2}$

Now the other roots are obtained by using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting n=0, 1, 2, ----------------------------

Graph:



Ex: Using Newton Raphson method find the root of an equation $x^3 - 4x - 9 = 0$ correct to three decimal places.

Ans: Given $x^3 - 4x - 9 = 0$

Let $f(x) = x^3 - 4x - 9$,$f'(x) = 3x^2 - 4$

$f(0) = -9(-ve)$

$f(1) = -12(-ve)$

$f(2) = -9(-ve)$

$f(3) = 6(+ve)$

So the roots lies between 2 and 3

Let $x_0 = \frac{2+3}{2} = 2.5$

Now

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^3 - 4x_n - 9}{3{x_n}^2 - 4} = \frac{3{x_n}^3 - 4x_n - {x_n}^3 + 4x_n + 9}{3{x_n}^2 - 4} = \frac{2{x_n}^3 + 9}{3{x_n}^2 - 4}$$

Put n=0, $x_1 = \frac{2{x_0}^3 + 9}{3{x_0}^2 - 4} = \frac{2(2.5)^3 + 9}{3(2.5)^2 - 4} = 2.729$

n=1, $x_2 = \frac{2{x_1}^3 + 9}{3{x_1}^2 - 4} = \frac{2(2.729)^3 + 9}{3(2.729)^2 - 4} = 2.707$

n=2, $x_3 = \frac{2{x_2}^3 + 9}{3{x_2}^2 - 4} = \frac{2(2.707)^3 + 9}{3(2.707)^2 - 4} = 2.707$

Hence the root is 2.707

Assignment:

1.Find the root of an equation $x^3 - 5x + 1 = 0$ *using bisection method correct to*

Two decimal places, which lies between 2 and 3.

2.Using Newton's method find the root of an equation $x^3 - 3x + 1 = 0$ correct to three decimal places.

3.Using Newton's method find the root of an equation $x^3 - 6x + 4 = 0$ correct to three decimal places.

4. Using Newton's method find the square root of 28 correct to three decimal places.

5. Find the root of an equation $x^3 - x - 11 = 0$ *using bisection method correct to* three decimal places.

6.Using Newton's Raphson method Find the cube root of 41 correct to four decimal places.

7. Using Newton's Raphson method Find the square root of 32 correct to four decimal places.

CHAPTER -2
FINITE DIFFERENCE & INTERPOLATION

FINITE DIFFERENCES AND INTERPOLATION

In Engineering , sometimes it is required to evaluate a function *f(x)* at some argument (independent variable) from a given set of tabulated values of *f(x)* within some interval. This can be done by studying a new concept called as Interpolation. Before studying interpolation, one should have an idea on the finite differences which is being used in interpolation.

The Finite Differences are:

1. **Forward Differences**
2. **Backward Differences**
3. **Central Differences**
4. **Average Differences**

If $y=f(x)$ is tabulated at equally spaced points x_0, $x_1=x_0+h$, $x_2=x_0+2h$, …. $x_n=x_0+nh$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, ….. $y_n=f(x_n)$ respectively, then the different type of differences are defined by :

A) FORWARD DIFFERENCES:

1st forward differences:

$\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$,…… etc

$\Delta y_i = y_{i+1} - y_i$, $i = 0,1,2,...n-1$ are called as 1st forward differences.

2nd forward differences:

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$,....*etc.*

$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$, $i = 0,1,2,....n-1$

are called as 2nd forward differences .

3rd forward differences:

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$, $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$, …. etc

are called as 3rd forward differences.

Similarly, the other higher order differences can be found.

TABLE OF FORWARD DIFFERENCES (For n=5)

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
x_1	y_1	Δy_0				
x_2	y_2	Δy_1	$\Delta^2 y_0$			
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$		
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	
x_5	y_5	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$

Example-1

Construct the forward difference table for $y = 3x^4 - x + 5$,given x= -1,1,2,3,4 and hence find the value of $\Delta f(2), \Delta^2 f(1)$ .Then Find the leading term and leading differences in the table.

x	y	Δ	Δ^2	Δ^3	Δ^4
-1	9				
1	7	-2			
2	51	44	46		
3	245	194	150	104	
4	769	524	330	180	76

From the table, $\Delta f(2) = \Delta y_2 = 194$, $\Delta^2 f(1) = \Delta^2 y_1 = 150$

B) BACKWARD DIFFERENCES

1st backward differences:

$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, ……etc.

$\nabla y_{i+1} = y_{i+1} - y_i$, $i = 0,1,2,...n-1$ are called as 1st backward differences.

2nd backward differences:

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$,....*etc.*

$\nabla^2 y_{i+1} = \nabla y_{i+1} - \nabla y_i$, $i = 1,2,....$

are called as 2nd backward differences .

3rd backward differences:

$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$, $\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$, …. etc

are called as 3rd backward differences.

Similarly, the other higher order differences can be found.

TABLE OF BACKWARD DIFFERENCES (FOR n=5)

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
x_1	y_1	∇y_1				
x_2	y_2	∇y_2	$\nabla^2 y_2$			
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

C) SHIFT OPERATOR

This operator is denoted by E and the inverse shift operator is denoted by E^{-1}.
If *f(x)* be any function and h = interval of differencing.
Then,

$$Ef(x) = f(x+h),$$
$$E^2 f(x) = E(E(f(x))) = E(f(x+h)) = f(x+2h),$$
$$E^n f(x) = f(x+nh), \qquad n = 1,2,3.....$$
$$E^{-1} f(x) = f(x-h),$$
$$E^{-2} f(x) = f(x-2h)$$
$$E^{-n} f(x) = f(x-nh), \qquad n = 1,2,3,......$$

Example-2

Evaluate: $E\tan x - E^{-2}e^x$, taking h=1

Answer:

$$E\tan x - E^{-2}e^{x}$$
$$= \tan(x+h) - e^{x-2h}$$
$$= \tan(x+1) - e^{x-2}$$

FORMULA:

1. Taylor's series expansion:

$$f(x+h) = f(x) + h\frac{f'(x)}{1!} + h^2\frac{f''(x)}{2!} + h^3\frac{f'''(x)}{3!} + \dots\dots\dots$$

2. Exponential Series:

$$e^{x} = 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x^3}{3!} + \dots.$$

RELATION BETWEEN THE OPERATORS

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $E = e^{hD}, D = \frac{d}{dx}$
4. $E^{-1} = e^{-hD}, D = \frac{d}{dx}$

Proof:

1. For any function *f(x)*

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - 1.f(x)$$
$$= (E-1)f(x)$$
$$\Rightarrow \Delta = E - 1$$

2. Similar to 1.

3.

$$Ef(x) = f(x+h) = f(x) + h\frac{f'(x)}{1!} + h^2\frac{f''(x)}{2!} + h^3\frac{f'''(x)}{3!} + \dots\dots\dots$$
$$= f(x) + \frac{hDf(x)}{1!} + \frac{h^2D^2f(x)}{2!} + \frac{h^3D^3f(x)}{3!} + \dots\dots.$$
$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \dots.\right)f(x)$$
$$= e^{hD}f(x)$$
$$\Rightarrow E = e^{hD}$$

4. Proceed as in 3.

Assignments:

1. Evaluate the following:

a) $\frac{\Delta}{E}\sin x$, b) $\frac{\Delta^2}{E}e^{x}, h=1$ c) $(\Delta+\nabla)^2 x^2$,

2. Prove that:

a) $(1+\Delta)(1-\nabla)=1$, b) $\Delta\nabla=\nabla\Delta=\Delta+\nabla$

Note:

1. **If a polynomial is of degree n, then n^{th} order difference is constant and other higher order differences will be 0.**
2. **If we are given (n+1) values of y in a data, then we can find a polynomial of degree n.**

<u>Example-6</u>

Find the missing values in the data:

1.

x	1	1.2	1.4	1.6	1.8	2	2.2	2.4
Y=f(x)	0	0.182		0.47	0.587	-	0.788	0.875

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1	**0**						
1.2	**0.182**	**0.182**					
1.4	**A**	**A-0.18**	**A-0.364**				
1.6	**0.47**	**0.47-A**	**0.652-2A**	**1.016-3A**			
1.8	**0.587**	**0.117**	**A-0.353**	**3A-1.005**	**6A-2.021**		
2.0	**B**	**B-0.58**	**B-0.704**	**B-A-0.351**	**B-3A+0.65**	**B-9A+2.675**	
2.2	**0.788**	**0.788-I**	**1.375-2B**	**2.079-3B**	**2.43-4B+A**	**-5B+4A+1.776**	**-6B+13A-0.899**
2.4	**0.875**	**0.087**	**B-0.701**	**3B-2.076**	**6B-4.155**	**10B-A-6.585**	**15B-5A-8.361**

Since, in the data we are given only six values of y , so therefore we can find a polynomial of degree 5.Hence the 6^{th} differences will be 0.

So,

-6B+13A-0.899=0 -------- (1)

15B-5A-8.361=0 ---------- (2)

Solving above equations we get A=21.217/55 and B=113.188/165

The missing values are A=0.386 and B=0.686 approximately.

Assignment:

Find the missing data in the tables:

1.

x	1	2	3	4	5
y	10	17		31	45

2.

x	1	1.5	2	2.5	3	3.5	4	4.5
y	9	-	13	21	37	53	-	82

3.

x	1	3	5	7	9
y	9	-	17	22	25

INTERPOLATION

If $y=f(x)$ is tabulated at equally spaced points x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, …..$y_n=f(x_n)$ respectively, then the process of getting the values of y at some intermediate value between x_0, and x_n is called as Interpolation and that of getting the values of y at some x which is outside of x_0 and x_n is called as Extrapolation.

Various Interpolation formula are:

1. Newton's Forward Interpolation
2. Newton's Backward Interpolation
3. Lagrange's Interpolation

1. Newton's Forward Interpolation Formula:

If $y=f(x)$ is tabulated at (n+1) equally spaced points x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, …..$y_n=f(x_n)$ respectively, then the Newton's Forward Interpolation Formula is given by:

$$y_p = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + ... + p(p-1)(p-2)....(p-n+1)\frac{\Delta^n y_0}{n!}$$

which is a polynomial of degree n. Here, $p = \frac{x - x_0}{h}$.

2. Newton's Backward Interpolation Formula:

If $y=f(x)$ is tabulated at (n+1) equally spaced points x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, …..$y_n=f(x_n)$ respectively, then the Newton's Backward Interpolation Formula is given by:

$$y_p = y_n + p\nabla y_n + p(p+1)\frac{\nabla^2 y_n}{2!} + p(p+1)(p+2)\frac{\nabla^3 y_n}{3!} + ... + p(p+1)(p+2)....(p+n-1)\frac{\nabla^n y_n}{n!}$$

which is a polynomial of degree n. Here, $p = \frac{x - x_n}{h}$.

3. Lagrange's Interpolation Formula:

If $y = f(x)$ is tabulated at (n+1) points x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh$ (not necessarily equally spaced) as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, …..$y_n=f(x_n)$ respectively, then the Lagrange's Interpolation Formula is given by:

$$y = l_0(x)\times y_0 + l_1(x)\times y_1 + ... + l_n(x)\times y_n$$

Where $l_0(x) = \dfrac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}$, $l_1(x) = \dfrac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)}$, ……………..

$$l_n(x) = \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).....(x_n-x_{n-1})}$$

Notes:

1. Newton's forward Interpolation method is used to find the value of y at a point x which is given near the beginning of the data.(the arguments should be equally spaced)
2. Newton's backward Interpolation method is used to find the value of y at a point x which is given near the end of the data.(the arguments should be equally spaced)
3. Lagrange's Interpolation method is used to find the value of y at any point x.

Examples:-

1. Construct a 3rd degree Newton's Forward Interpolating polynomial using the data:

x	1	2	3	4
y	10	17	23	32

Answer :-

X	Y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1	**10**			
2	17	**7**		
3	23	6	**-1**	
4	32	9	3	**4**

From the above table $h=1,\ x_0=1,\ y_0=10,\ \Delta y_0=7,\ \Delta^2 y_0=-1,\ \Delta^3 y_0=4,\ p=\dfrac{(x-x_0)}{h}=\dfrac{x-1}{1}=x-1$

Newton's Forward Interpolation Formula is given by:

$$y_p = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} \qquad ------(1)$$

which is a polynomial of degree 3.

Putting all the values in equation (1) we have,

$$y_p(x) = 10 + (x-1)\times 7 + (x-1)(x-1-1)\frac{-1}{2!} + (x-1)(x-1-1)(x-1-2)\frac{4}{3!}$$

$$= 10 + 7(x-1) - \frac{(x-1)(x-2)}{2} + 4(x-1)(x-2)(x-3)$$

2. Compute y at x = 3 using the data given below:

x	2	4	6	8	10
y	15	23	34	47	59

Answer :-

X	Y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
2	**15**				
4	23	**8**			
6	34	11	**3**		
8	47	13	2	**-1**	
10	59	12	-1	-3	**-2**

We have to compute y at x = 2.5, therefore

From the above table we take,

$$h = 2 \ x_0 = 2, \ y_0 = 15, \ \Delta y_0 = 8, \ \Delta^2 y_0 = 3, \ \Delta^3 y_0 = -1, \ \Delta^4 y_0 = -2, \ p = \frac{(x - x_0)}{h} = \frac{2.5 - 2}{2} = 0.25$$

Newton's Forward Interpolation Formula is given by:

$$y_p = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + p(p-1)(p-2)(p-3)\frac{\Delta^4 y_0}{4!} \quad ------(1)$$

Putting all the values in equation (1) we have,

$$y(2.5) = 15 + 0.25 \times 8 + 0.25(0.25-1)\frac{3}{2!} + 0.25 \times (0.25-1)(0.25-2)\frac{-1}{3!} + 0.25 \times (0.25-1) \times (0.25-2) \times (0.25-3)\frac{-2}{4!}$$

$$= 15 + 2 - \frac{9}{32} - 0.0546875 \ + 0.0752 \ = 17.0752 - 0.3359375 \ \cong 16.74$$

3. Construct Newton's Backward Interpolating polynomial using the data :

x	-1	0	1	2	3
y	9	21	42	63	87

and hence find y for x= 2.5.

Solution:

X	Y	∇	∇^2	∇^3	∇^4
-1	**9**				
0	21	**12**			
1	42	21	**9**		
2	63	21	0	**-9**	
3	87	24	3	**3**	**12**

$$p = \frac{x - x_n}{h} = \frac{x-3}{1} = x-3$$

$$y_p = y_4 + p\nabla y_4 + p(p+1)\frac{\nabla^2 y_4}{2!} + p(p+1)(p+2)\frac{\nabla^3 y_4}{3!} + p(p+1)(p+2)(p+3)\frac{\nabla^4 y_4}{4!}$$

$$=87 + (x-3)24 + (x-3)(x-2)\frac{3}{2} + (x-3)(x-2)(x-1)\frac{3}{6} + (x-3)(x-2)(x-1)x\frac{12}{24}$$

To find y at x=2.5 take $x_n = 2, p = \frac{x - x_n}{h} = \frac{2.5-2}{1} = 0.5$

$$y_p = 87 + 0.5\times 24 + 0.5\times 1.5\times \frac{3}{2} + 0.5\times 1.5\times 2.5\times \frac{3}{6} + 0.5\times 1.5\times 2.5\times 3.5\times \frac{12}{24}$$

= ?

4. Construct Lagrange's Interpolating polynomial using the data:

X	1	2	3
Y	42	63	87

and hence find y for x= 2.5.

Ans. $l_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2}$

$$l_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{(x-1)(x-3)}{-1}$$

$$l_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{(x-1)(x-2)}{2}$$

$$y(x) = l_0(x)\times y_0 + l_1(x)\times y_1 + l_2(x)\times y_2$$

$$= \frac{(x-2)(x-3)}{2}\times 42 + \frac{(x-1)(x-3)}{-1}\times 63 + \frac{(x-1)(x-2)}{2}\times 87$$

$$= simplify$$

$$y(2.5) = l_0(2.5)\times y_0 + l_1(2.5)\times y_1 + l_2(2.5)\times y_2$$

$$= \frac{(2.5-2)(2.5-3)}{2}\times 42 + \frac{(2.5-1)(2.5-3)}{-1}\times 63 + \frac{(2.5-1)(2.5-2)}{2}\times 87$$

$$= simplify$$

Numerical Integration

We can evaluate the definite integrals of the type: $\int_a^b f(x)dx$ provided we know the integration $\int f(x)dx$. But it is always not possible to evaluate all the definite integrals. In that case, we can implement a new method as discussed below.

Definition: Suppose a function y=f(x) is continuous in the interval [a, b] and tabulated at equally spaced points a= x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh=b$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$, …..$y_n=f(x_n)$, then the process of evaluating the approximate value of the integral $\int_a^b f(x)dx$ is called as Numerical Integration.

Techniques of Numerical Integration:

1. Newton-Cote's Rule
2. Trapezoidal Rule
3. Simpson's 1/3rd Rule

1. Newton-Cote's Rule

If y=f(x) is continuous in the interval [a, b] and tabulated at equally spaced points a= x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh=b$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$,…..$y_n=f(x_n)$, then we can construct the Newton's Forward Interpolating Polynomial as:

$$y_p = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + ... + p(p-1)(p-2)....(p-n+1)\frac{\Delta^n y_0}{n!} \text{ ------ (1)}$$

which is a polynomial of degree n. Here $p = \frac{x-x_0}{h}$.

Now, $\int_a^b f(x)dx$ can be evaluated by integrating the function approximated by the interpolating polynomial in Eq. (1) over [a, b].

$$\int_a^b f(x)dx = \int_a^b (y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + ..)dx$$

$$= \int_0^n (y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!} + ..)ndp \text{ as } p = \frac{x-x_0}{h} \Rightarrow hp = x - x_0 \Rightarrow dx = hdp$$

$$= \left[n(y_0 p + \frac{p^2}{2}\Delta y_0 + (\frac{p^3}{3} - \frac{p^2}{2})\frac{\Delta^2 y_0}{2!} + ...) \right]_0^n = n^2(y_0 + \frac{n}{2}\Delta y_0 + (\frac{n^2}{3} - \frac{n}{2})\frac{\Delta^2 y_0}{2!} + ...)$$

2. Trapezoidal Rule

Formula:

If y=f(x) is continuous in the interval [a, b] and tabulated at equally spaced points a= x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh=b$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$,…..$y_n=f(x_n)$, then the approximate value of $\int_a^b f(x)dx$ will be

$$\int_a^b f(x)dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + ...) + y_n\right]$$

Here, h = (b-a)/n. This is called as trapezoidal formula.

Example:

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ **using Trapezoidal rule taking h=1/4 . Hence find an approximate value of π.**

Answer: Here a=0, b=1, h=1/4=0.25

¼ = (1-0)/n

⇒n=1/(1/4)=4.

X	X_0=0	X_1=0.25	X_2=0.5	X_3=0.75	X_4=1
y	1=y_0	0.9411=y_1	0.8=y_2	0.64=y_3	0.5=y_4

$$\int_a^b f(x)dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + ...) + y_n\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{2}\left[1 + 2(0.9411 + 0.8 + 0.64) + 0.5\right]$$

$$= 0.782775$$

To find the value of **π**

$$\int_0^1 \frac{1}{1+x^2}dx = 0.782775$$

$$\Rightarrow \left[\tan^{-1} x\right]_0^1 = 0.782775$$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} 0 = 0.782775$$

$$\Rightarrow \frac{\pi}{4} = 0.782775$$

$$\Rightarrow \pi = 4 \times 0.782775 = 3.1311$$

Assignment:

i) Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ **using Trapezoidal rule taking h=1/5 . Hence find an approximate value ofπ.**

ii) Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ **using Trapezoidal rule taking h=1/10. Hence find an approximate value ofπ.**

3. Simpson's 1/3rd rule:

Formula:

If y=f(x) is continuous in the interval [a, b] and tabulated at equally spaced points a= x_0, $x_1=x_0+h$, $x_2=x_0+2h$, ….$x_n=x_0+nh=b$ as $y_0=f(x_0)$, $y_1=f(x_1)$, $y_2=f(x_2)$,…..$y_n=f(x_n)$, then the approximate value of $\int_a^b f(x)dx$ will be

$$\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n\right]$$

Here, h = (b-a)/n and n is always **even**. This is called as **Simpson's 1/3rd rule**.

Example:

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ **using Simpson's 1/3rd rule taking 11 ordinates. Hence find an approximate value of π.**

Answer:

Here a=0, b=1, n=10. h=(b-a)/n=(1-0)/10=0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	1	0.99	0.961	0.917	0.862	0.8	0.735	0.671	0.609	0.552	0.5

$$\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n\right]$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{0.1}{3}\left[1 + 4(0.99 + 0.917 + 0.8 + 0.671 + 0.552) + 2(0.961 + 0.862 + 0.735 + 0.609) + 0.5\right]$$

$$\approx 0.78513$$

<u>To find the value of **π**</u>

$$\int_0^1 \frac{1}{1+x^2}dx = 0.78513$$

$$\Rightarrow \left[\tan^{-1} x\right]_0^1 = 0.78513$$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} 0 = 0.78513$$

$$\Rightarrow \frac{\pi}{4} = 0.78513$$

$$\Rightarrow \pi = 4 \times 0.78513 = 3.14052$$

Example:

Evaluate $\int_0^{\pi/2} \sqrt{\cos x}\,dx$ **using Simpson's 1/3rd rule taking h=π/12(or 7 ordinates). (Keep your calculator in "rad" mode)**

Answer:

Here a=0, b= $\pi/2$, n=6.

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12} = \frac{\pi}{6}$	$\frac{3\pi}{12} = \frac{\pi}{4}$	$\frac{4\pi}{12} = \frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12} = \frac{\pi}{2}$

y	1	0.98281	0.93061	0.84089	0.70711	0.50874	0

$$\int_a^b f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n]$$

$$\int_0^1 \sqrt{\cos x}dx = \frac{\frac{\pi}{12}}{3}[1 + 4(0.98281 + 0.84089 + 0.50874) + 2(0.93061 + 0.70711) + 0] = 1.18723 \quad \text{(ans)}$$

Assignments

i. Evaluate $\int_0^1 x^3 dx$ using a) Trapezoidal rule b) Simpson's 1/3rd rule taking a suitable h.

ii. Evaluate $\int_0^5 \frac{dx}{4x+5}$ using a) Trapezoidal rule b) Simpson's 1/3rd rule considering 10 sub intervals.

iii. Evaluate $\int_0^1 \cos x dx$ using a) Trapezoidal rule taking 5 equal parts.

EXTRA QUESTIONS FOR PRACTICE (Chapterwise)

Numerical Integration :

1.Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using simpson's 1/3 rule.Hence obtain the approximate value of π *in each case*.

2.Evaluate $\int_0^6 \frac{dx}{1+x^2}$.(consider n=6)

3.Use Trapezoidal rule evaluate $\int_0^1 x^3 dx$ considering five sub intervals.

4.Using simpson's 1/3 rule evaluate $\int_1^2 \frac{dx}{x}$ taking h=1/4.

5. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ *taking four subintervals*.

6.Evaluate $\int_0^5 \frac{dx}{4x+5}$ using 11 ordinates.(ii ordinates means n=10).

7.Evaluate $\int_0^{\frac{\pi}{2}} sinx\ dx$ using 11 ordinates.

8. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{cosx}\ dx$.(Here consider n=6)

9.Evaluate $\int_0^1 \frac{dx}{1+x}$ (consider n=4).

Finite Difference and Interpolation:

1.Evaluate $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$

2.Evaluate $\Delta^n\left(\frac{1}{x}\right)\ taking\ h=1.$

3.Show that $\Delta \log f(x) = log\left[1+\frac{\Delta f(x)}{f(x)}\right]$

4.Prove that $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$

5. Prove that $\Delta^3 y_i = y_{i+3} + 3y_{i+2} - 3y_{i+1} - y_i$

6. By forming a difference table find the missing values in the following table assuming that the fourth differences are equal to zero.

X:	0	5	10	15	20	25
Y :	6	10	----	17	-----	31

7. Find the missing value of the following table

X:	0	1	2	3	4
Y:	1	3	9	--	81

8. Find the missing value of the following table

X:	0	1	2	3	4	5	6
Y:	5	11	22	40	----	140	-----

9.Evaluate $\Delta^n e^x$

Lagrange's Interpolation:

1.Using Lagrange's interpolation find the value of y at x=10 if

X:	5	6	9	11
Y:	12	13	14	16

2.Using Lagrange's Interpolation find f(9)

X:	5	7	11	13	17
F(x):	180	392	1452	2366	5202

3.Using Lagrange's interpolation find the value of y at x=5 if

Y(1)=8,y(3)=15,y(4)=11 ,y(8)=32 and y(10)=40

5.Using Lagrange's interpolation find the value of x at y=15 if

X:	5	6	9	11

Y: 12	13	14	16

6.Using Lagrange's interpolation find the value of x at f(x)=0 if

F(30)=-30,f(34)=-13,f(38)=3 and f(42)=18.

7.Find lagrange's interpolation polynomial that takes same values of y at the given points.

If y(1)=-3 , y(3)=9,y(4)=30 and y(6)=132.

Newton's Forward and Backward Interpolation:

1.Using Newton's forward interpolation find f(1.6) if

X:	1	1.4	1.8	2.2
F(x):	3.49	4.82	5.96	6.5

2.Using Newton's Backward Interpolation find f(1.28) if

F(1.15)=1.0723,f(1.20)=1.0954 ,f(1.25)=1.1180 and f(1.30)=1.1401

3.Estimate the value of f(22) and f(42) if

X:	20	25	30	35	40	45
Y:	354	332	291	260	231	204

4.Find the cubic polynomial which takes the following values

X:	0	1	2	3
Y:	1	2	1	10

Hence or otherwise find f(4).

5.Find the number of men getting wages between Rs.10 and Rs.15 from the following table

Wages in Rs. :	0-10	10-20	20-30	30-40
Frequency:	9	30	35	42

6.The area of a circle of diameter d is given for the following values

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Find the approximate values for the areas of circles of diameter 82 and 91 respectively.

Reference Books:

Higher engineering mathematics By Dr B.S. Grewal (khanna publishers)