Disclaimer

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MODULE#1

CHAPTER#1

1. Basic Concept of Control System

Control Engineering is concerned with techniques that are used to solve the following six problems in the most efficient manner possible.

(a)The identification problem :to measure the variables and convert data for analysis.

(b)The representation problem:to describe a system by an analytical form or mathematical model

(c)The solution problem:to determine the above system model response.

(d)The stability problem:general qualitative analysis of the system

(e)The design problem: modification of an existing system or develop a new one

(f)The optimization problem: from a variety of design to choose the best.

The two basic approaches to solve these six problems are conventional and modern approach. The electrical oriented conventional approach is based on complex function theory. The modern approach has mechanical orientation and based on the state variable theory.

Therefore, control engineering is not limited to any engineering discipline but is equally applicable to aeronautical, chemical, mechanical, environmental, civil and electrical engineering. For example, a control system often includes electrical, mechanical and chemical components. Furthermore, as the understanding of the dynamics of business, social and political systems increases; the ability to control these systems will also increase.

1.1. Basic terminologies in control system

System: A combination or arrangement of a number of different physical components to form a whole unit such that that combining unit performs to achieve a certain goal.

Control: The action to command, direct or regulate a system.

Plant or process: The part or component of a system that is required to be controlled.

Input: It is the signal or excitation supplied to a control system.

Output: It is the actual response obtained from the control system.

Controller: The part or component of a system that controls the plant.

Disturbances: The signal that has adverse effect on the performance of a control system.

Control system: A system that can command, direct or regulate itself or another system to achieve a certain goal.

Automation: The control of a process by automatic means

Control System: An interconnection of components forming a system configuration that will provide a desired response.

Actuator: It is the device that causes the process to provide the output. It is the device that provides the motive power to the process.

Design: The process of conceiving or inventing the forms, parts, and details of system to achieve a specified purpose.

Simulation: A model of a system that is used to investigate the behavior of a system by utilizing actual input signals**.**

Optimization: The adjustment of the parameters to achieve the most favorable or advantageous design.

Feedback Signal: A measure of the output of the system used for feedback to control the system.

Negative feedback: The output signal is feedback so that it subtracts from the input signal.

Block diagrams: Unidirectional, operational blocks that represent the transfer functions of the elements of the system.

Signal Flow Graph (SFG): A diagram that consists of nodes connected by several directed branches and that is a graphical representation of a set of linear relations.

Specifications: Statements that explicitly state what the device or product is to be and to do. It is also defined as a set of prescribed performance criteria.

Open-loop control system: A system that utilizes a device to control the process without using feedback. Thus the output has no effect upon the signal to the process.

Closed-loop feedback control system: A system that uses a measurement of the output and compares it with the desired output.

Regulator: The control system where the desired values of the controlled outputs are more or less fixed and the main problem is to reject disturbance effects.

Servo system: The control system where the outputs are mechanical quantities like acceleration, velocity or position.

Stability: It is a notion that describes whether the system will be able to follow the input command. In a non-rigorous sense, a system is said to be unstable if its output is out of control or increases without bound.

Multivariable Control System: A system with more than one input variable or more than one output variable.

Trade-off: The result of making a judgment about how much compromise must be made between conflicting criteria.

1.2. Classification

1.2.1. Natural control system and Man-made control system:

Natural control system: It is a control system that is created by nature, i.e. solar system, digestive system of any animal, etc.

Man-made control system: It is a control system that is created by humans, i.e. automobile, power plants etc.

1.2.2. Automatic control system and Combinational control system:

Automatic control system: It is a control system that is made by using basic theories from mathematics and engineering. This system mainly has sensors, actuators and responders.

Combinational control system: It is a control system that is a combination of natural and man-made control systems, i.e. driving a car etc.

1.2.3. Time-variant control system and Time-invariant control system:

Time-variant control system: It is a control system where any one or more parameters of the control system vary with time i.e. driving a vehicle.

Time-invariant control system: It is a control system where none of its parameters vary with time i.e. control system made up of inductors, capacitors and resistors only.

1.2.4. Linear control system and Non-linear control system:

Linear control system: It is a control system that satisfies properties of homogeneity and additive.

- Homogeneous property: $f(x+y)=f(x)+f(y)$
- Additive property: $f(\alpha x)=\alpha f(x)$

Non-linear control system: It is a control system that does not satisfy properties of homogeneity and additive, i.e. $f(x)=x^3$

1.2.5. Continuous-Time control system and Discrete-Time control system:

Continuous-Time control system: It is a control system where performances of all of its parameters are function of time, i.e. armature type speed control of motor.

Discrete -Time control system: It is a control system where performances of all of its parameters are function of discrete time i.e. microprocessor type speed control of motor.

1.2.6. Deterministic control system and Stochastic control system:

Deterministic control system: It is a control system where its output is predictable or repetitive for certain input signal or disturbance signal.

Stochastic control system: It is a control system where its output is unpredictable or non-repetitive for certain input signal or disturbance signal.

1.2.7. Lumped-parameter control system and Distributed-parameter control system:

Lumped-parameter control system: It is a control system where its mathematical model is represented by ordinary differential equations.

Distributed-parameter control system: It is a control system where its mathematical model is represented by an electrical network that is a combination of resistors, inductors and capacitors.

1.2.8. Single-input-single-output (SISO) control system and Multi-input-multi-output (MIMO) control system:

SISO control system: It is a control system that has only one input and one output.

MIMO control system: It is a control system that has only more than one input and more than one output.

1.2.9. Open-loop control system and Closed-loop control system:

Open-loop control system: It is a control system where its control action only depends on input signal and does not depend on its output response.

Closed-loop control system: It is a control system where its control action depends on both of its input signal and output response.

1.3. Open-loop control system and Closed-loop control system

1.3.1. Open-loop control system:

It is a control system where its control action only depends on input signal and does not depend on its output response as shown in Fig.1.1.

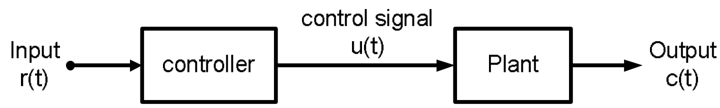


Fig.1.1. An open-loop system

Examples: traffic signal, washing machine, bread toaster, etc.

Advantages:

- Simple design and easy to construct
- Economical
- Easy for maintenance
- Highly stable operation

Dis-advantages:

- Not accurate and reliable when input or system parameters are variable in nature
- Recalibration of the parameters are required time to time

1.3.2. Closed-loop control system:

It is a control system where its control action depends on both of its input signal and output response as shown in Fig.1.2.

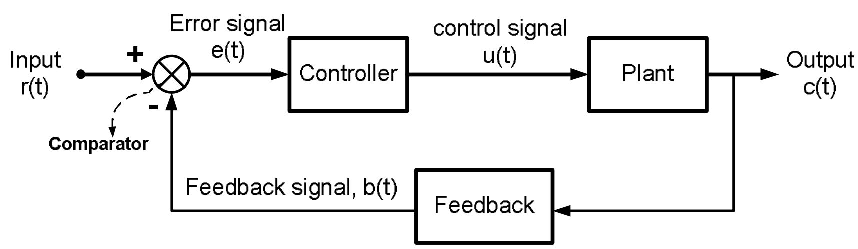


Fig.1.2. A closed-loop system

Examples: automatic electric iron, missile launcher, speed control of DC motor, etc.

Advantages:

- More accurate operation than that of open-loop control system
- Can operate efficiently when input or system parameters are variable in nature
- Less nonlinearity effect of these systems on output response
- High bandwidth of operation
- There is facility of automation
- Time to time recalibration of the parameters are not required

Dis-advantages:

- Complex design and difficult to construct

- Expensive than that of open-loop control system
- Complicate for maintenance
- Less stable operation than that of open-loop control system

1.3.3. Comparison between Open-loop and Closed-loop control systems:

It is a control system where its control action depends on both of its input signal and output response.

Sl. No.	Open-loop control systems	Closed-loop control systems
1	No feedback is given to the control system	A feedback is given to the control system
2	Cannot be intelligent	Intelligent controlling action
3	There is no possibility of undesirable system oscillation(hunting)	Closed loop control introduces the possibility of undesirable system oscillation(hunting)
4	The output will not very for a constant input, provided the system parameters remain unaltered	In the system the output may vary for a constant input, depending upon the feedback
5	System output variation due to variation in parameters of the system is greater and the output very in an uncontrolled way	System output variation due to variation in parameters of the system is less.
6	Error detection is not present	Error detection is present
7	Small bandwidth	Large bandwidth
8	More stable	Less stable or prone to instability
9	Affected by non-linearities	Not affected by non-linearities
10	Very sensitive in nature	Less sensitive to disturbances
11	Simple design	Complex design
12	Cheap	Costly

1.4. Servomechanism

It is the feedback unit used in a control system. In this system, the control variable is a mechanical signal such as position, velocity or acceleration. Here, the output signal is directly fed to the comparator as the feedback signal, b(t) of the closed-loop control system. This type of system is used where both the command and output signals are mechanical in nature. A position control system as shown in Fig.1.3 is a simple example of this type mechanism. The block diagram of the servomechanism of an automatic steering system is shown in Fig.1.4.

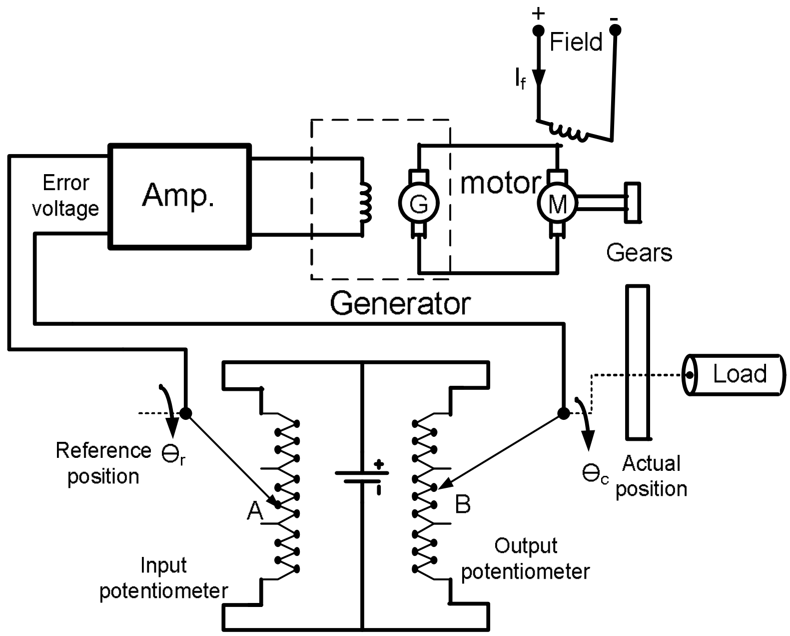


Fig.1.3. Schematic diagram of a servomechanism

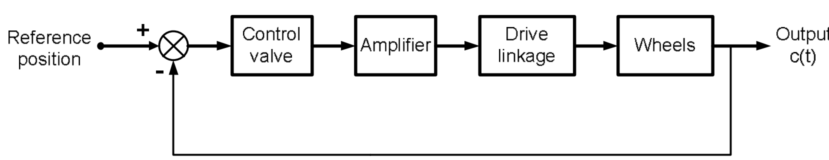


Fig.1.4. Block diagram of a servomechanism

Examples:

- Missile launcher
- Machine tool position control
- Power steering for an automobile
- Roll stabilization in ships, etc.

1.5. Regulators

It is also a feedback unit used in a control system like servomechanism. But, the output is kept constant at its desired value. The schematic diagram of a regulating system is shown in Fig.1.5. Its corresponding simplified block diagram model is shown in Fig.1.6.

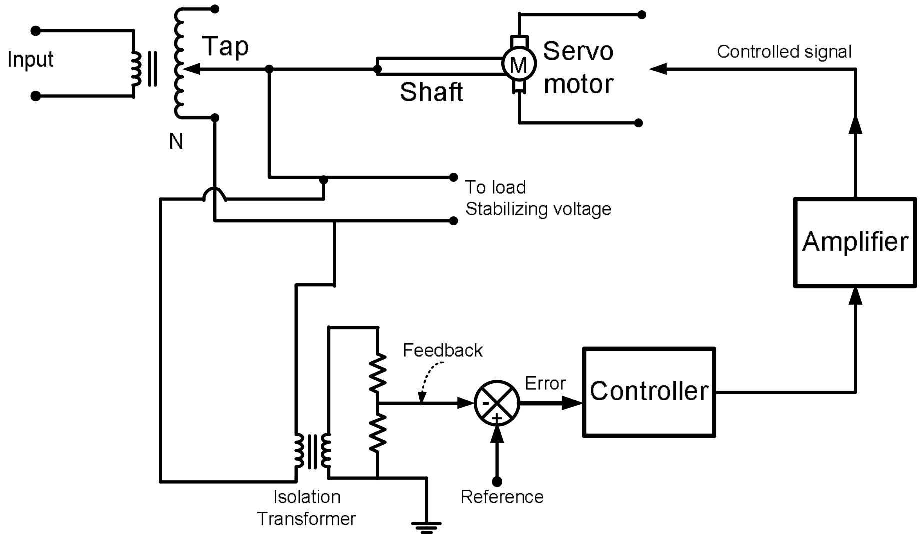


Fig.1.5. Schematic diagram of a regulating system

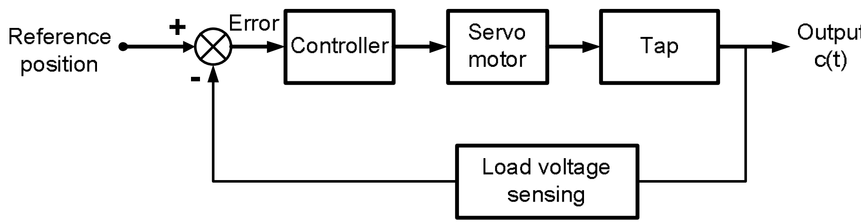


Fig.1.6. Block diagram of a regulating system

Examples:

- Temperature regulator
- Speed governor
- Frequency regulators, etc.

CHAPTER#2

2. Control System Dynamics

2.1. Definition: It is the study of characteristics behaviour of dynamic system, i.e.

(a) Differential equation

i. First-order systems
ii. Second-order systems

(b) System transfer function: Laplace transform

2.2. Laplace Transform: Laplace transforms convert differential equations into algebraic equations. They are related to frequency response.

$$\mathrm{L}\left\{x(t)\right\} = X(s) = \int_0^\infty x(t)e^{-st}dt \tag{2.1}$$

$$\mathrm{L}\left\{x(t)\right\} = X(s) = \int_0^\infty x(t)e^{-st}dt \tag{2.2}$$

No.	Function	Time-domain x(t)= $\mathcal{L}^{-1}$\{X(s)\}	Laplace domain X(s)= $\mathcal{L}$\{x(t)\}
1	Delay	$\delta(t-\tau)$	$e^{-\tau s}$
2	Unit impulse	$\delta(t)$	1
3	Unit step	u(t)	$\frac{1}{s}$
4	Ramp	t	$\frac{1}{s^2}$
5	Exponential decay	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
6	Exponential approach	$\left(1-e^{-\alpha t}\right)$	$\frac{\alpha}{s(s+\alpha)}$
7	Sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
8	Cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
9	Hyperbolic sine	$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
10	Hyperbolic cosine	$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$
11	Exponentially decaying sine wave	$e^{-\alpha t}\sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
12	Exponentially decaying cosine wave	$e^{-\alpha t}\cos \omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

2.3. Solution of system dynamics in Laplace form: Laplace transforms can be solved using partial fraction method.

A system is usually represented by following dynamic equation.

$$N(s) = \frac{A(s)}{B(s)} \tag{2.3}$$

The factor of denominator, B(s) is represented by following forms,

i. Unrepeated factors

ii. Repeated factors
iii. Unrepeated complex factors

(i) Unrepeated factors

$$\begin{aligned}\frac{N(s)}{(s+a)(s+b)} &= \frac{A}{s+a}+\frac{B}{s+b} \\ &= \frac{A(s+b)+B(s+a)}{(s+a)(s+b)}\end{aligned} \tag{2.4}$$

By equating both sides, determine A and B.

Example 2.1:

Expand the following equation of Laplacetransform in terms of its partial fractionsand obtain its time-domain response.

$$Y(s)=\frac{2s}{(s+1)(s+2)}$$

Solution:

The following equation in Laplacetransform is expandedwith its partial fractions as follows.

$$\frac{2s}{(s+1)(s+2)}=\frac{A}{(s+1)}+\frac{B}{(s+2)}$$

$$\Rightarrow \frac{2s}{(s+1)(s+2)}=\frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

By equating both sides, A and B are determined as $A=-2, B=4$. Therefore,

$$Y(s)=-\frac{2}{(s+1)}+\frac{4}{(s+2)}$$

Taking Laplace inverse of above equation,

$$y(t)=-2e^{-t}+4e^{-2t}$$

(ii) Unrepeated factors

$$\frac{N(s)}{(s+a)^2}=\frac{A}{(s+a)^2}+\frac{B}{(s+a)}=\frac{A+B(s+a)}{(s+a)^2} \tag{2.5}$$

By equating both sides, determine A and B.

Example 2.2:

Expand the following equation of Laplacetransform in terms of its partial fractionsand obtain its time-domain response.

$$Y(s)=\frac{2s}{(s+1)^2(s+2)}$$

Solution:

The following equation in Laplacetransform is expandedwith its partial fractions as follows.

$$\frac{2s}{(s+1)^2(s+2)}=\frac{A}{(s+1)^2}+\frac{B}{(s+1)}+\frac{C}{(s+2)}$$

By equating both sides, A and B are determined as $A=-2, B=4$. Therefore,

$$Y(s)=-\frac{2}{(s+1)^2}+\frac{4}{(s+1)}-\frac{4}{(s+2)}$$

Taking Laplace inverse of above equation,

$$y(t)=-2te^{-t}+4e^{-t}-4e^{-2t}$$

(iii) Complex factors: They contain conjugate pairs in the denominator.

$$\frac{N(s)}{(s+a)(s+\overline{a})}=\frac{As+B}{(s+\alpha)^2+\beta^2} \tag{2.6}$$

By equating both sides, determine A and B.

Example 2.3:

Expand the following equation of Laplacetransform in terms of its partial fractionsand obtain its time-domain response.

$$Y(s)=\frac{2s+1}{(s+1+j)(s+1-j)}$$

Solution:

The following equation in Laplacetransform is expandedwith its partial fractions as follows.

$$Y(s)=\frac{2s}{(s+1)^2+1}+\frac{1}{(s+1)^2+1}$$

Taking Laplace inverse of above equation,

$$y(t)=2e^{-t}\cos t+e^{-t}\sin t$$

2.4. Initial value theorem:

$$\lim_{t\to 0}[y(t)]=\lim_{s\to\infty}\left[sY(s)\right] \tag{2.7}$$

Example 2.4:

Determine the initial value of the time-domain response of the following equation using the initial-value theorem.

$$Y(s)=\frac{2s+1}{(s+1+j)(s+1-j)}$$

Solution:

Solution of above equation,

$$y(t)=2e^{-t}\cos t+e^{-t}\sin t$$

Applying initial value theorem,

$$\lim_{s\to\infty}\frac{s(2s+1)}{(s+1+j)(s+1-j)}=2$$

2.5. Final value theorem:

$$\lim_{t\to\infty}(y(t))=\lim_{s\to 0}\left[sY(s)\right] \tag{2.8}$$

Example 2.5:

Determine the initial value of the time-domain response of the following equation using the initial-value theorem.

$$Y(s)=\frac{2s}{(s+1)^2(s+2)}$$

Solution:

Solution of above equation,

$$y(t) = -2te^{-t} + 4e^{-t} - 4e^{-2t}$$

Applying final value theorem,

$$\lim_{s \to \infty} \frac{s(2s+1)}{(s+1+j)(s+1-j)} = 2$$

CHAPTER#3

3. Transfer Function

3.1. Definition: It is the ratio of Laplace transform of output signal to Laplace transform of input signal assuming all the initial conditions to be zero, i.e.

Let, there is a given system with input r(t) and output c(t) as shown in Fig.3.1 (a), then its Laplace domain is shown in Fig.3.1 (b). Here, input and output are R(s) and C(s) respectively.

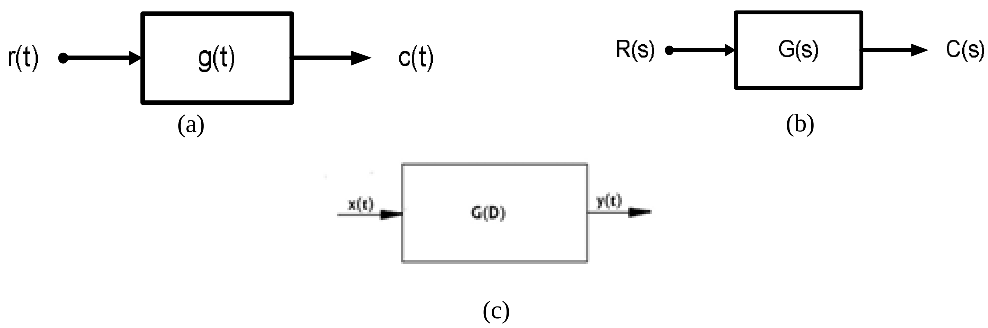


Fig.3.1. (a) A system in time domain, (b) a system in frequency domainand (c) transfer function with differential operator

G(s) is the transfer function of the system. It can be mathematically represented as follows.

$$G(s) = \left. \frac{C(s)}{R(s)} \right|_{zero \quad initial \quad condition} \qquad \text{Equation Section (Next)}(3.1)$$

Example 3.1: Determine the transfer function of the system shown inFig.3.2.

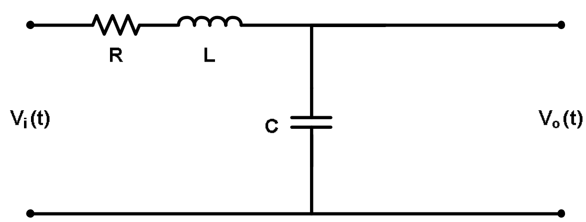


Fig.3.2. a system in time domain

Solution:

Fig.3.1 is redrawn in frequency domain as shown in Fig.3.2.

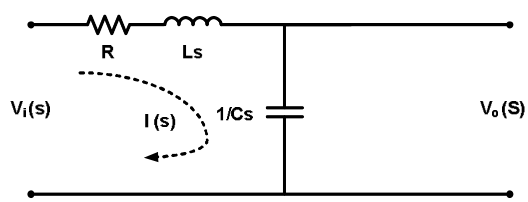


Fig.3.2. a system in frequency domain

Applying KVL to loop-1 of the Fig.3.2

$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s) \tag{3.2}$$

Applying KVL to loop-2 of the Fig.3.2

$$V_o(s) = \left(\frac{1}{Cs}\right) I(s) \tag{3.3}$$

From eq (2.12),

$$I(s) = V_o(s) / \left(\frac{1}{Cs}\right) = CsV_o(s) \tag{3.4}$$

Now, using eq (2.13) in eq (2.10),

$$\begin{aligned} V_i(s) &= \left(R + Ls + \frac{1}{Cs}\right) CsV_o(s) \\ \Rightarrow \frac{V_o(s)}{V_i(s)} &= \frac{1}{\left(R + Ls + \frac{1}{Cs}\right)Cs} = \frac{1}{LCs^2 + RCs + 1} \end{aligned} \tag{3.5}$$

Then transfer function of the given system is

$$G(s) = \frac{1}{LCs^2 + RCs + 1} \tag{3.6}$$

3.2. General Form of Transfer Function

$$G(s) = \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - z_j)} \tag{3.7}$$

Where, $z_1,\ z_2 \ldots z_m$ are called zeros and $p_1,\ p_2 \ldots p_n$ are called poles.

Number of poles *n* will always be greater than the number of zeros *m*

Example 3.2:

Obtain the pole-zero map of the following transfer function.

$$G(s) = \frac{(s-2)(s+2+j4)(s+2-j4)}{(s-3)(s-4)(s-5)(s+1+j5)(s+1-j5)}$$

Solution:

The following equation in Laplacetransform is expandedwith its partial fractions as follows.

Zeros	Poles
s=2	s=3
s=-2-j4	s=4
s=-2+j4	s=5

	s=-1-j5
	s=-1+j5

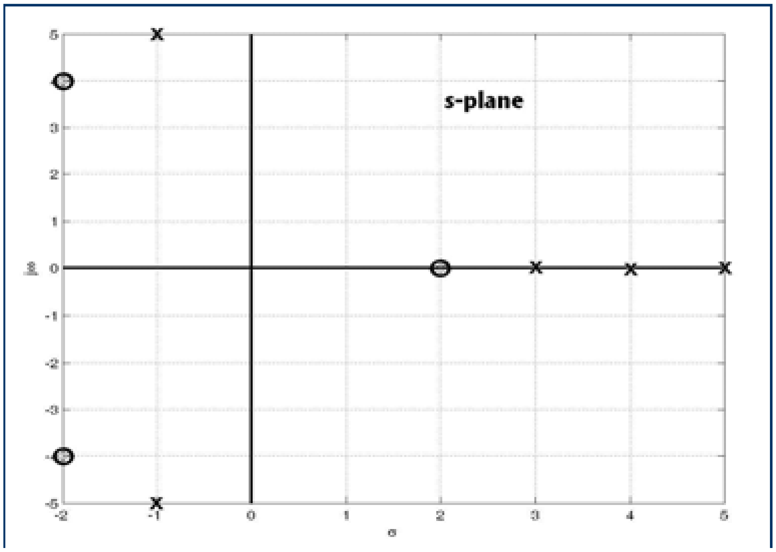


Fig.3.3. pole-zero map

3.3. Properties of Transfer function:

- Zero initial condition
- It is same as Laplace transform of its impulse response
- Replacing 's' by $\frac{d}{dt}$ in the transfer function, the differential equation can be obtained
- Poles and zeros can be obtained from the transfer function
- Stability can be known
- Can be applicable to linear system only

3.4. Advantages of Transfer function:

- It is a mathematical model and gain of the system
- Replacing 's' by $\frac{d}{dt}$ in the transfer function, the differential equation can be obtained
- Poles and zeros can be obtained from the transfer function
- Stability can be known
- Impulse response can be found

3.5. Disadvantages of Transfer function:

- Applicable only to linear system
- Not applicable if initial condition cannot be neglected
- It gives no information about the actual structure of a physical system

CHAPTER#4

4. Description of physical system

4.1. Components of a mechanical system: Mechanical systems are of two types, i.e. (i) translational mechanical system and (ii) rotational mechanical system.

4.1.1. Translational mechanical system

There are three basic elements in a translational mechanical system, i.e. (a) mass, (b) spring and (c) damper.

(a) **Mass:** A mass is denoted by M. If a force f is applied on it and it displays distance x, then $f = M\frac{d^2x}{dt^2}$ as shown in Fig.4.1.

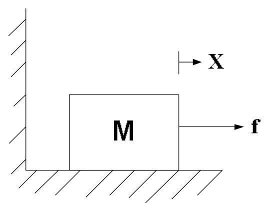


Fig.4.1. Force applied on a mass with displacement in one direction

If a force f is applied on a massM and it displays distance $x_1$in the direction of f and distance x_2 in the opposite direction, then $f = M\left(\frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2}\right)$ as shown in Fig.4.2.

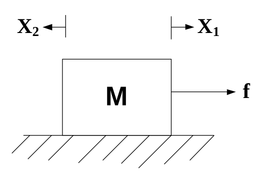


Fig.4.2. Force applied on a mass with displacement two directions

(b) **Spring:** A spring is denoted by K. If a force f is applied on it and it displays distance x, then $f = Kx$ as shown in Fig.4.3.

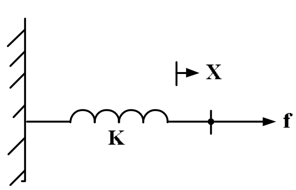


Fig.4.3. Force applied on a spring with displacement in one direction

If a force f is applied on a springK and it displays distance $x_1$in the direction of f and distance x_2 in the opposite direction, then $f = K(x_1 - x_2)$ as shown in Fig.4.4.

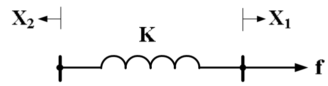


Fig.4.4. Force applied on a spring with displacement in two directions

(c) **Damper:** A damper is denoted by D. If a force *f* is applied on it and it displays distance *x*, then $f = D\frac{dx}{dt}$ as shown in Fig.4.5.

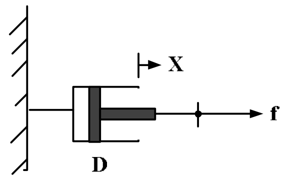


Fig.4.5. Force applied on a damper with displacement in one direction

If a force *f* is applied on a damperD and it displays distance $x_1$in the direction of *f* and distance x_2 in the opposite direction, then $f = D\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$ as shown in Fig.4.6.

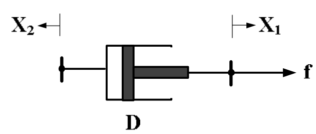


Fig.4.6. Force applied on a damper with displacement in two directions

4.1.2. Rotational mechanical system

There are three basic elements in a Rotational mechanical system, i.e. (a) inertia, (b) spring and (c) damper.

(a) **Inertia:** A body with aninertia is denoted by *J*. If a torque*T* is applied on it and it displays distanceΘ, then $T = J\frac{d^2\theta}{dt^2}$. If a torque*T* is applied on a body with inertia *J* and it displays distance Θ_1 in the direction of *T* and distance Θ_2 in the opposite direction, then $T = J\left(\frac{d^2\theta_1}{dt^2} - \frac{d^2\theta_2}{dt^2}\right)$.

(b) **Spring:** A spring is denoted by K. If a torque*T* is applied on it and it displays distanceΘ, then $T = K\theta$. If a torque*T* is applied on a body with inertia *J* and it displays distance Θ_1 in the direction of *T* and distance Θ_2 in the opposite direction, then $T = K(\theta_1 - \theta_2)$.

(c) **Damper:** A damper is denoted by D. If a torque*T* is applied on it and it displays distanceΘ, then $T = D\frac{d\theta}{dt}$. If a torque*T* is applied on a body with inertia *J* and it displays distance Θ_1 in the direction of T and distance Θ_2 in the opposite direction, then $T = D\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right)$.

4.2. Components of an electrical system: There are three basic elements in an electrical system, i.e. (a) resistor (R), (b) inductor(L) and (c) capacitor (C). Electrical systems are of two types, i.e. (i) voltage source electrical system and (ii) current source electrical system.

4.2.1. Voltage source electrical system: If i is the current through a resistor(Fig.4.7) and v is the voltage drop in it, then $v = Ri$.

If i is the current through an inductor (Fig.4.7) and v is the voltage developed in it, then $v = L\frac{di}{dt}$.

If i is the current through a capacitor(Fig.4.7) and v is the voltage developed in it, then $v = \frac{1}{C}\int idt$.

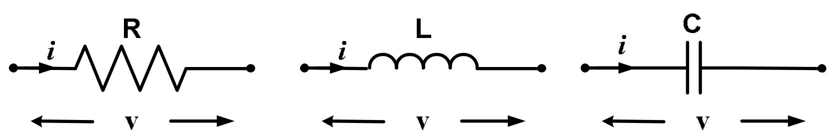


Fig.4.7. Current and voltage shown in resistor, inductor and capacitor

4.2.2. Current source electrical system:

If i is the current through a resistor and v is the voltage drop in it, then $i = \frac{v}{R}$.

If i is the current through an inductor and v is the voltage developed in it, then $i = \frac{1}{L}\int vdt$.

If i is the current through a capacitor and v is the voltage developed in it, then $i = C\frac{dv}{dt}$.

4.2.3. Work out problems:

Q.4.1. Find system transfer function betweenvoltage drop across the capacitanceand input voltage in the followingRC circuit as shown in Fig.4.8.

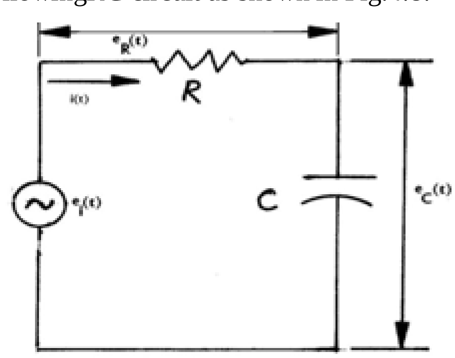


Fig.4.8.

Solution

Voltage across resistance, $e_R(t) = i(t)R$

Voltage across capacitance, $e_C(t) = \frac{1}{C}\int i(t)dt$

Total voltage drop, $e_i = e_R + e_C = i(t)R + \frac{1}{C}\int i(t)dt$

Laplace transform of above equation, $E_i(s) = I(s)\left(R + \frac{1}{Cs}\right)$

System transfer function betweenvoltage drop across the capacitanceand input voltage, $\frac{E_C(s)}{E_i(s)} = \frac{1}{RCs+1} = \frac{1}{\tau s+1}$

where, $RC = \tau$ is the time-constant

Q.4.2. Find system transfer function betweenfunction between the inductance currentto the source currentin the followingRL circuit as shown in Fig.4.9.

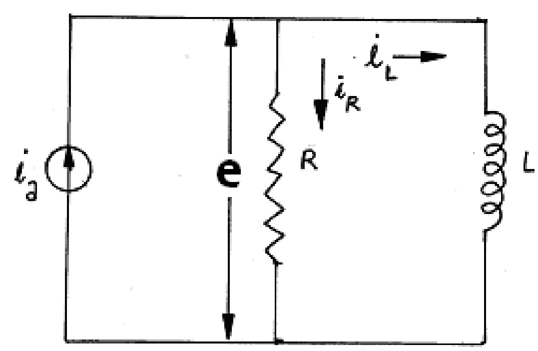


Fig.4.9.

Voltage across the Resistance, $e(t) = i_R R \Rightarrow i_R = \frac{e(t)}{R}$

Voltage across the Inductance, $e(t) = L\frac{di_L}{dt} \Rightarrow i_L = \frac{1}{L}\int e(t)dt$

Total current, $i_a = i_R + i_L = \frac{e(t)}{R} + \frac{1}{L}\int e(t)dt$

Laplace transform of the current source,

$I_a(s) = E(s)\left(\frac{1}{R} + \frac{1}{Ls}\right)$ and $I_L(s) = \frac{E}{Ls}$

Transfer function between the inductance current to the source current,

$$\frac{I_L(s)}{I_a(s)} = \frac{1}{\frac{L}{R}s+1} = \frac{1}{\tau s+1}$$

where $\tau = \dfrac{L}{R}$ is the time-constant

Q.4.3. Find system transfer function betweenfunction between the capacitance voltageto the source voltage in the followingRLC circuit as shown in Fig.4.10.

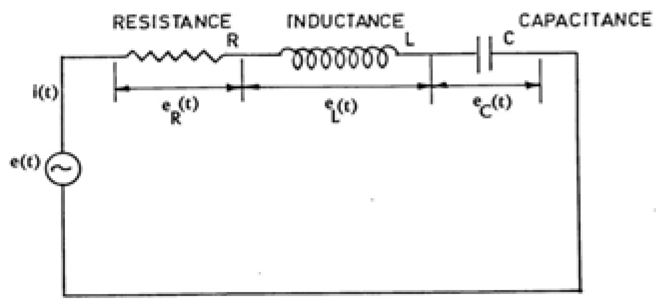


Fig.4.10.

Voltage across the Resistance, $e_R(t) = iR$

Voltage across the Inductance, $e_L(t) = L\dfrac{di}{dt}$

Voltage across thecapacitance, $e_C(t) = \dfrac{1}{C}\int idt$

Total voltage, $e(t) = iR + L\dfrac{di}{dt} + \dfrac{1}{C}\int idt$

Laplace transform of the voltage source, $E(s) = I(s)\left(R + Ls + \dfrac{1}{Cs}\right)$

Transfer function between capacitance voltage and source voltage

$$\frac{E_C(s)}{E(s)} = \frac{1}{Cs\left(R + Ls + \dfrac{1}{Cs}\right)} = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

where $\omega_n = \dfrac{1}{\sqrt{LC}}$ and $\zeta = \dfrac{R}{2\sqrt{\dfrac{L}{C}}}$

Q.4.4. Find the transfer function of the following Spring-mass-damperas shown in Fig.4.11.

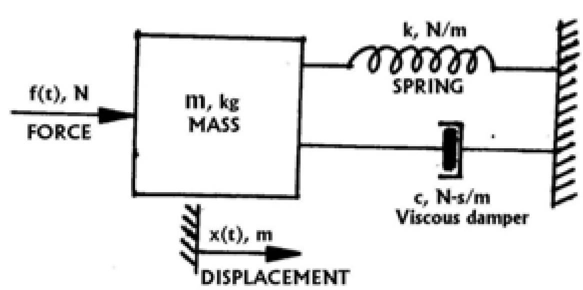
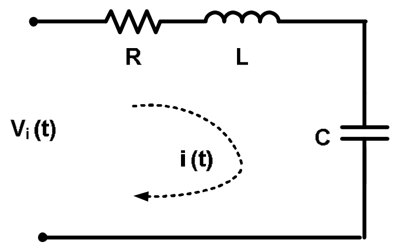


Fig.4.11.

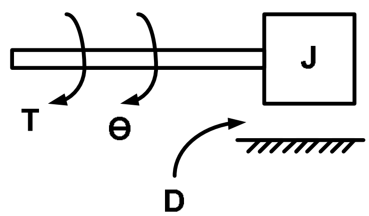
Solution

$$\frac{X(s)}{F(s)}=\frac{1}{ms^2+cs+k}=\frac{1}{m\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

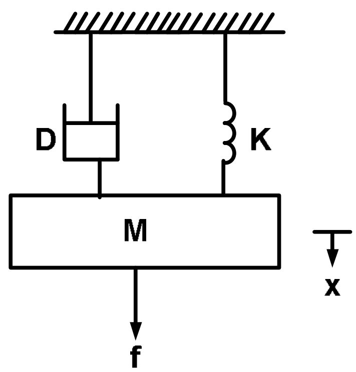
4.3. Analogous system: Fig.4.12 shows a translational mechanical system, a rotational control system and a voltage-source electrical system.



(a)



(b)



(c)

Fig.4.12. (a) a voltage-source electrical system,(b) a translational mechanical system and (c) a rotational control system

From Fig4.12 (a), (b) and (c), we have

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=v(t)$$
$$J\frac{d^2\theta}{dt^2}+D\frac{d\theta}{dt}+K\theta=T \qquad \textbf{Equation Chapter 8 Section 0}(4.1)$$
$$M\frac{d^2x}{dt^2}+D\frac{dx}{dt}+Kx=f$$

Where,

$$q = \int idt \tag{4.2}$$

The solutions for all the above three equations given by eq (4.2) are same. Therefore, the above shown three figures are analogous to each other. There are two important types of analogous systems, i.e. force-voltage (f-v) analogy and force-current analogy. From eq (4.2), f-v analogy can be drawn as follows.

Translational	Rotational	Electrical
Force (f)	Torque (T)	Voltage (v)
Mass (M)	Inertia (J)	Inductance (L)
Damper (D)	Damper (D)	Resistance (R)
Spring (K)	Spring (K)	Elastance (1/C)
Displacement (x)	Displacement (Θ)	Charge (q)
Velocity (u) = $\dot{x}$	Velocity (u) = $\dot{\theta}$	Current (i) = $\dot{q}$

Similarly, f-i analogy that can be obtainedfrom eq (4.1), can be drawn as follows.

Translational	Rotational	Electrical
Force (f)	Torque (T)	Current (i)
Mass (M)	Inertia (J)	Capacitance (C)
Damper (D)	Damper (D)	Conductance (1/R)
Spring (K)	Spring (K)	Reciprocal of Inductance (1/L)
Displacement (x)	Displacement (Θ)	Flux linkage (ψ)
Velocity (u) = $\dot{x}$	Velocity (u) = $\dot{\theta}$	Voltage (v) = $\dot{\psi}$

4.4. Mathematical model of armature controlled DC motor: The armature control type speed control system of a DC motor is shown in Fig.4.6. The following components are used in this system.

R_a=resistance of armature

L_a=inductance of armature winding

i_a=armature current

I_f=field current

E_a=applied armature voltage

E_b=back emf

T_m=torque developed by motor

Θ=angular displacement of motor shaft

J=equivalent moment of inertia and load referred to motor shaft

f=equivalent viscous friction coefficient of motor and load referred to motor shaft

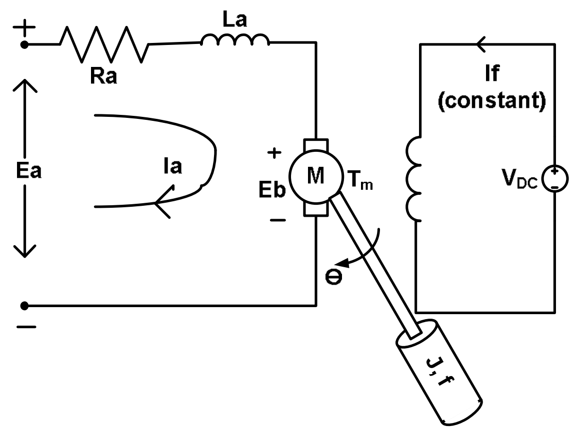


Fig.4.6. Schematic diagram of armature control type speed control system of a DC motor

The air-gap flux ϕ is proportional of the field current i.e.

$$\phi = K_f I_f \tag{4.3}$$

The torque T_m developed by the motor is proportional to the product of armature current and air gap flux i.e.

$$T_m = k_1 K_f I_f i_a \tag{4.4}$$

In armature-controlled D.C. motor,the field current is kept constant,so that eq(4.4) can be written as follows.

$$T_m = K_t i_a \tag{4.5}$$

The motor back emf being proportional to speed is given as follows.

$$E_b = K_b \left(\frac{d\theta}{dt} \right) \tag{4.6}$$

The differential equation of the armature circuit is

$$L_a \left(\frac{di_a}{dt} \right) + R_a i_a + E_b = E_a \tag{4.7}$$

The torque equation is

$$J \left(\frac{d^2\theta}{dt^2} \right) + f \left(\frac{d\theta}{dt} \right) = T_m = K_t I_a \tag{4.8}$$

Taking the Laplace transforms of equations (4.6), (4.7) and (4.8), assuming zero initial conditions, we get

$$E_b(s) = sK_b \theta(s) \tag{4.9}$$

$$(sL_a + R_a) I_a(s) = E_a(s) - E_b(s) \tag{4.10}$$

$$(s^2 J + sf)\theta(s) = T_m(s) = K_t I_a \tag{4.11}$$

From eq(4.9) to (4.11) the transfer function of the system is obtained as,

$$G(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_t}{s\left[(R_a + sL_a)(sJ + f) + K_t K_b\right]} \tag{4.12}$$

Eq(4.12) can be rewritten as

$$G(s) = \frac{\theta(s)}{E_a(s)} = \left[\frac{\frac{K_t}{(R_a + sL_a)(sJ + f)}}{1 + \frac{K_t K_b}{(R_a + sL_a)(sJ + f)}}\right]\frac{1}{s} \tag{4.13}$$

The block diagram that is constructed from eq (4.13) is shown in Fig.4.7.

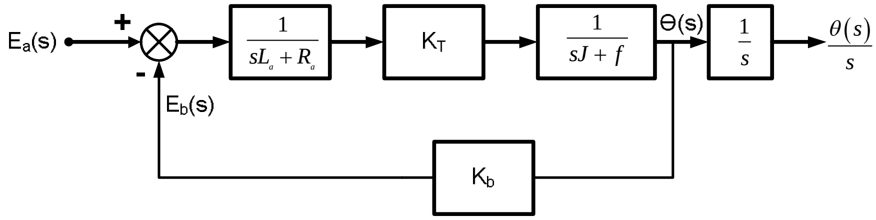


Fig.4.7. Block diagram of armature control type speed control system of a DC motor

The armature circuit inductance L_a is usually negligible. Therefore, eq(4.13) can be simplifiedas follows.

$$\frac{\theta(s)}{E_a(s)} = s^2\left(\frac{K_t}{R_a}\right)J + s\left(f + \frac{K_t K_b}{R_a}\right) \tag{4.14}$$

The term $\left(f + \frac{K_t K_b}{R_a}\right)$ indicates that the back emf of the motor effectively increases the viscous friction of the system. Let,

$$f' = f + \frac{K_t K_b}{R_a} \tag{4.15}$$

Where f' be the effective viscous friction coefficient. The transfer function given by eq(4.15) may be written in the following form.

$$\frac{\theta(s)}{E_a(s)} = \frac{K_m}{s(s\tau + 1)} \tag{4.16}$$

Here $K_m = \frac{K_t}{R_a f}$ = motor gain constant, and $\tau = \frac{J}{f'}$ = motor time constant.Therefore, the motor torque and back emf constant K_t, K_b are interrelated.

4.5. Mathematical model of field controlled DC motor: The field control type speed control system of a DC motor is shown in Fig.4.8. The following components are used in this system.

R_f=Field winding resistance

L_f=inductance of field winding

I_f=field current

e_f=field control voltage

T_m=torque developed by motor

Θ=angular displacement of motor shaft

J=equivalent moment of inertia and load referred to motor shaft

f=equivalent viscous friction coefficient of motor and load referred to motor shaft

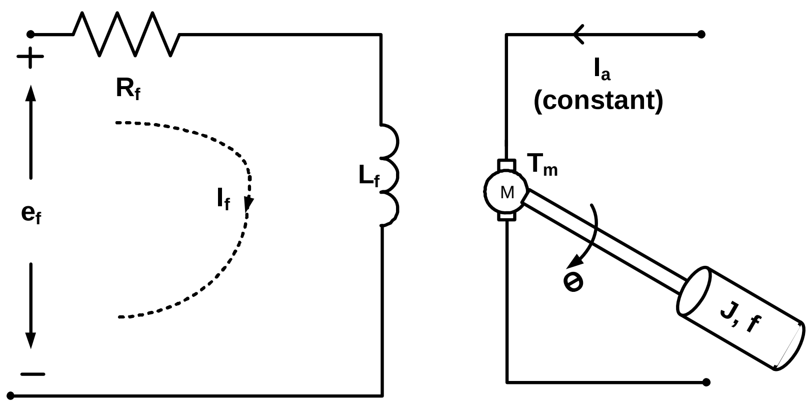


Fig.4.8. Block diagram of field control type speed control system of a DC motor

In field control motor the armature current is fed from a constant current source.The air-gap flux Φ is proportional of the field current i.e.

$$\phi = K_f I_f \tag{4.17}$$

The torque Tm developed by the motor is proportional to the product of armature current and air gap flux i.e.

$$T_m = k_1 K_f I_f I_a = K_t I_f \tag{4.18}$$

The equation for the field circuit is

$$L_f \frac{dI_f}{dt} + R_f I_f = E_f \tag{4.19}$$

The torque equation is

$$J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt} = T_m = K_t I_f \tag{4.20}$$

Taking the Laplace transforms of equations (4.19) and (4.20) assuming zero initial conditions, we get the following equations

$$\left(L_f s + R_f\right) I_f(s) = E_f(s) \tag{4.21}$$

and

$$\left(Js^2 + fs\right)\theta(s) = T_m(s) = K_t I_f(s) \tag{4.22}$$

From eq(4.21) and (4.22) the transfer function of the system is obtained as

$$G(s) = \frac{\theta(s)}{E_f(s)} = \frac{K_t}{s\left(R_f + sL_f\right)\left(Js + f\right)} \tag{4.23}$$

The transfer function given by eq(4.23) may be written in the following form.

$$\frac{\theta(s)}{E_a(s)} = \frac{K_t}{s\left(L_f s + R_f\right)\left(Js + f\right)} = \frac{K_m}{s\left(s\tau + 1\right)\left(s\tau' + 1\right)} \tag{4.24}$$

Here $K_m = \frac{K_t}{R_f f}$ = motor gain constant, and $\tau = \frac{L_f}{R_f}$ = time constant of field circuit and $\tau' = \frac{J}{f}$ = mechanical time constant.For small size motors field control is advantageous.The block diagram that is constructed from eq (4.24) is shown in Fig.4.9.

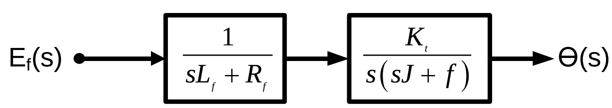


Fig.4.9. Block diagram of field control type speed control system of a DC motor

CHAPTER#5

5. Block Diagram Algebra

5.1. Basic Definition in Block Diagram model:

Block diagram: It is the pictorial representation of the cause-and-response relationship between input and output of a physical system.

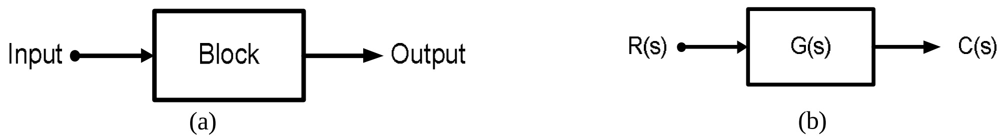


Fig.5.1. (a) A block diagram representation of a system and (b) A block diagram representation with gain of a system

Output: The value of input multiplied by the gain of the system.

$$C(s) = G(s)R(s) \tag{5.1}$$

Summing point: It is the component of a block diagram model at which two or more signals can be added or subtracted. In Fig.15, inputs R(s) and B(s) have been given to a summing point and its output signal is E(s). Here,

$$E(s) = R(s) - B(s) \tag{5.2}$$

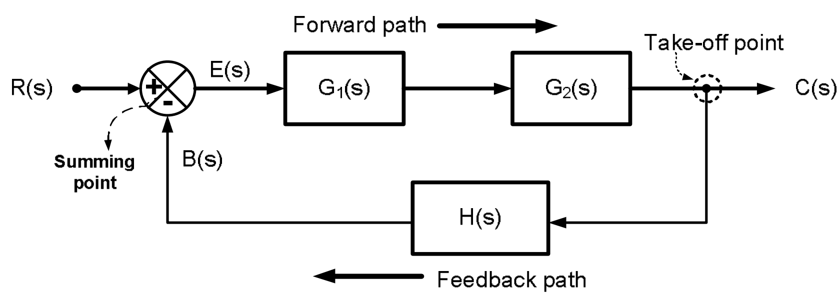


Fig.5.2. A block diagram representation of a systemshowing its different components

Take-off point: It is the component of a block diagram model at which a signal can be taken directly and supplied to one or more points as shown in Fig.5.2.

Forward path: It is the direction of signal flow from input towards output.

Feedback path: It is the direction of signal flow from output towards input.

5.2. Developing Block Diagram model from mathematical model:

Let's discuss this concept with the following example.

Example: A system is described by following mathematical equations. Find its corresponding block diagram model.

$$\dot{x}_1 = 3x_1 + 2x_2 + 5x_3 \tag{5.3}$$

$$\dot{x}_2 = x_1 + 4x_2 + 3x_3 \tag{5.4}$$

$$\dot{x}_3 = 2x_1 + x_2 + x_3 \tag{5.5}$$

Example: Eq (5.3), (5.4) and (5.5) are combiningly results in the following block diagram model.

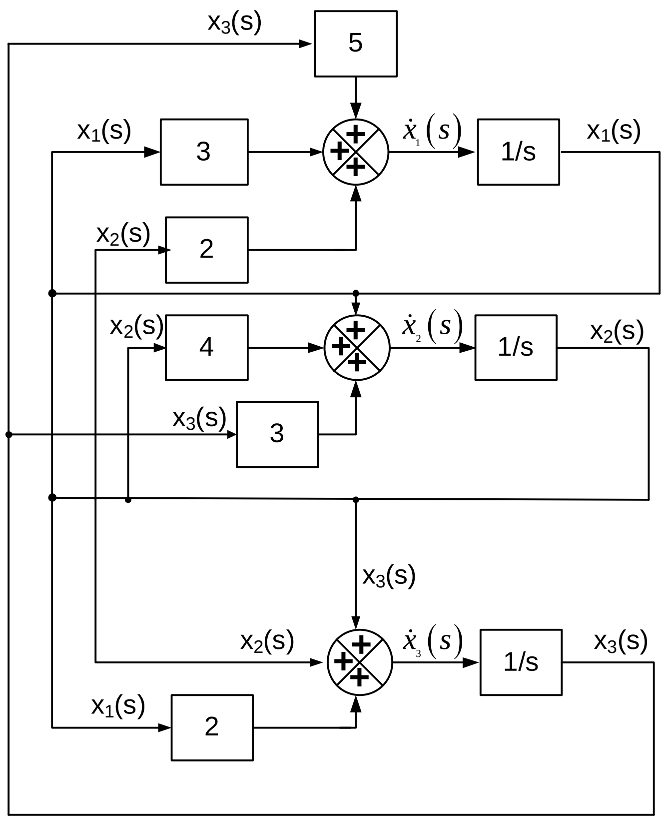
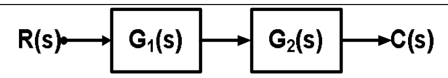
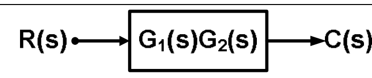
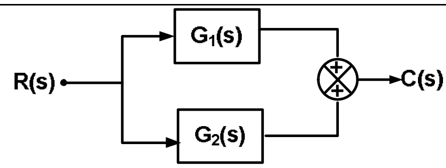
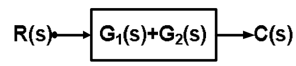
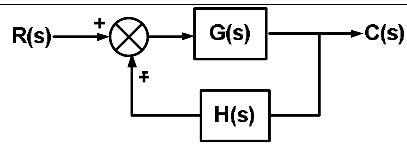
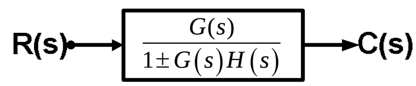
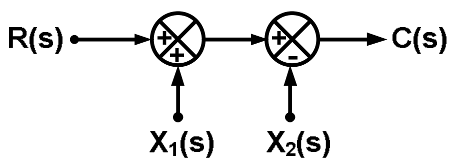
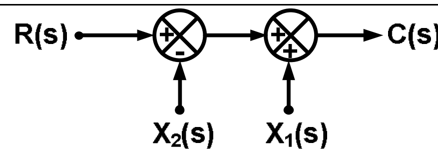
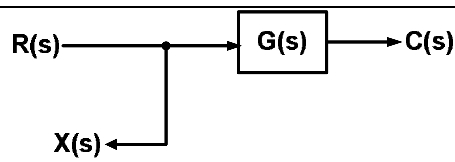
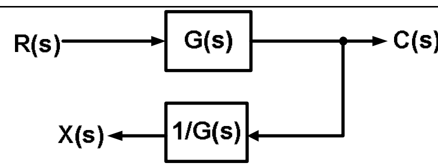
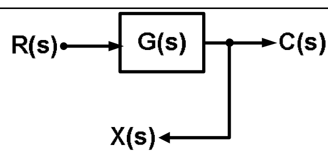
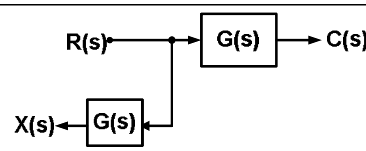
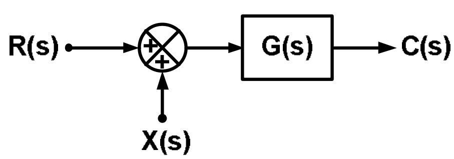
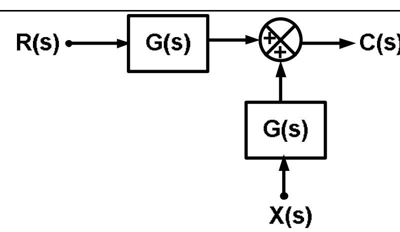
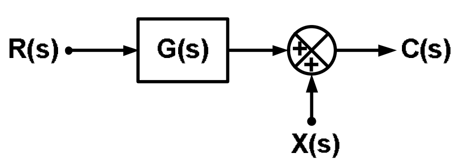
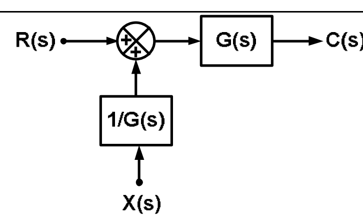


Fig.5.3. A block diagram representation of the above example

5.3. Rules for reduction of Block Diagram model:

Sl. No.	Rule No.	Configuration	Equivalent	Name
1	Rule 1			Cascade
2	Rule 2			Parallel
3	Rule 3			Loop
4	Rule 4			Associative Law
5	Rule 5			Move take-off point after a block
6	Rule 6			Move take-off point before a block
7	Rule 7			Move summing-point point after a block
8	Rule 8			Move summing-point point before a block

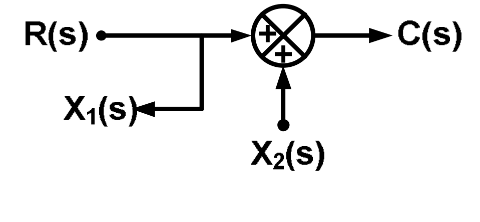
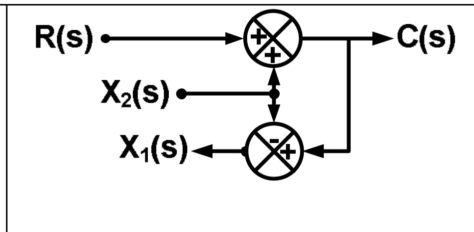
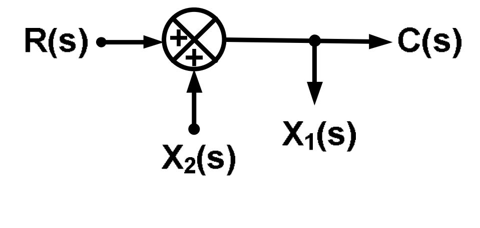
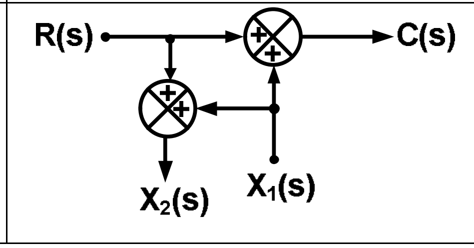
9	**Rule 9**			Move take-off point after a summing-point
10	**Rule 10**			Move take-off point before a summing-point

Fig.5.4. Rules for reduction of Block Diagram model

5.4. Procedure for reduction of Block Diagram model:

Step 1: Reduce the cascade blocks.

Step 2: Reduce the parallel blocks.

Step 3: Reduce the internal feedback loops.

Step 4: Shift take-off points towards right and summing points towards left.

Step 5: Repeat step 1 to step 4 until the simple form is obtained.

Step 6: Find transfer function of whole system as $\frac{C(s)}{R(s)}$.

5.5. Procedure for finding output of Block Diagram model with multiple inputs:

Step 1: Consider one input taking rest of the inputs zero, find output using the procedure described in section 4.3.

Step 2: Follow step 1 for each inputs of the given Block Diagram model and find their corresponding outputs.

Step 3: Find the resultant output by adding all individual outputs.

CHAPTER#6

6. Signal Flow Graphs (SFGs)

It is a pictorial representation of a system that graphically displays the signal transmission in it.

6.1. Basic Definitions in SFGs:

Input or source node: It is a node that has only outgoing branches i.e. node ‘r’ in Fig.6.1.

Output or sink node: It is a node that has only incoming branches i.e. node ‘c’ in Fig.6.1.

Chain node: It is a node that has both incoming and outgoing branches i.e. nodes ‘x_1’, ‘x_2’,‘x_3’,‘x_4’,‘x_5’and ‘x_6’ in Fig.6.1.

Gain or transmittance: It is the relationship between variables denoted by two nodes or value of a branch. In Fig.6.1, transmittances are ‘t_1’, ‘t_2’,‘t_3’,‘t_4’,‘t_5’and ‘t_6’.

Forward path: It is a path from input node to output node without repeating any of the nodes in between them. In Fig.6.1, there are two forward paths, i.e. path-1:‘r-x_1-x_2-x_3-x_4-x_5-x_6-c’ and path-2:‘r-x_1-x_3-x_4-x_5-x_6-c’.

Feedback path: It is a path from output node or a node near output node to a node near input node without repeating any of the nodes in between them (Fig.6.1).

Loop: It is a closed path that starts from one node and reaches the same node after trading through other nodes. In Fig.6.1, there are four loops, i.e. loop-1:‘x_2-x_3-x_4-x_1’, loop-2:‘x_5-x_6-x_5’, loop-3:‘x_1-x_2-x_3-x_4-x_5-x_6-x_1’ and loop-4:‘x_1-x_3-x_4-x_5-x_6-x_1’.

Self Loop: It is a loop that starts from one node and reaches the same node without trading through other nodes i.e. loop in node ‘x_4’ with transmittance ‘t_{55}’ in Fig.6.1.

Path gain: It is the product of gains or transmittances of all branches of a forward path. In Fig.6.1, the path gains are $P_1 = t_1t_2t_3t_4t_5$ (for path-1) and $P_2 = t_9t_3t_4t_5$ (for path-2).

Loop gain: It is the product of gains or transmittances of all branches of a loop In Fig.6.1, there are four loops, i.e. $L_1 = -t_2t_3t_6$, $L_2 = -t_5t_7$, $L_3 = -t_1t_2t_3t_4t_5t_8$, and $L_4 = -t_9t_3t_4t_5t_8$.

Dummy node: If the first node is not an input node and/or the last node is not an output node than a node is connected before the existing first node and a node is connected after the existing last node with unity transmittances. These nodes are called dummy nodes. In Fig.6.1, ‘r’ and ‘c’ are the dummy nodes.

Non-touching Loops: Two or more loops are non-touching loops if they don’t have any common nodes between them. In Fig.6.1, L_1 and L_2 are non-touching loops

Example:

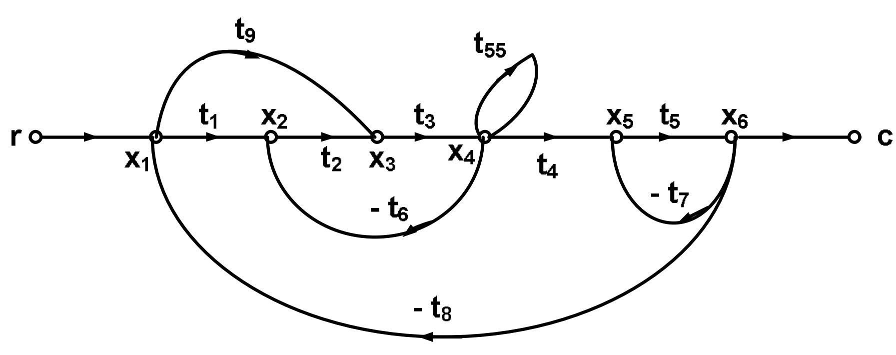


Fig.6.1. Example of a SFG model

6.2. Properties SFGs:

- Applied to linear system
- Arrow indicates signal flow
- Nodes represent variables, summing points and take-off points
- Algebraic sum of all incoming signals and outgoing nodes is zero
- SFG of a system is not unique
- Overall gain of an SFG can be determined by using Mason's gain formula

6.3. SFG from block diagram model:

Let's find the SFG of following block diagram model shown in Fig.6.2.

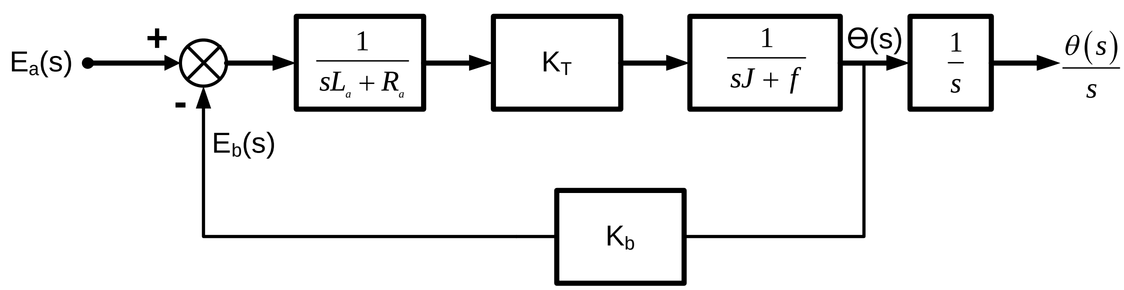


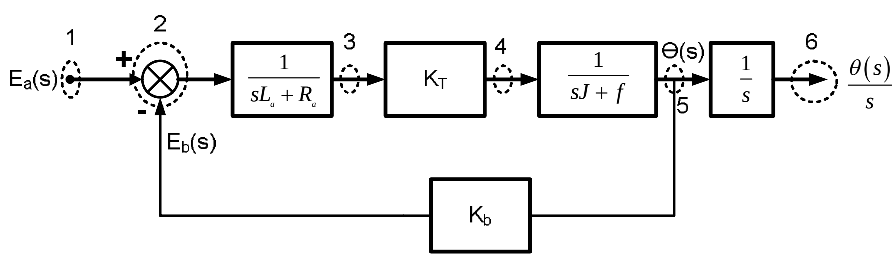
Fig.6.2. Armature type speed control of a DC motor

Step-1: All variables and signals are replaced by nodes.

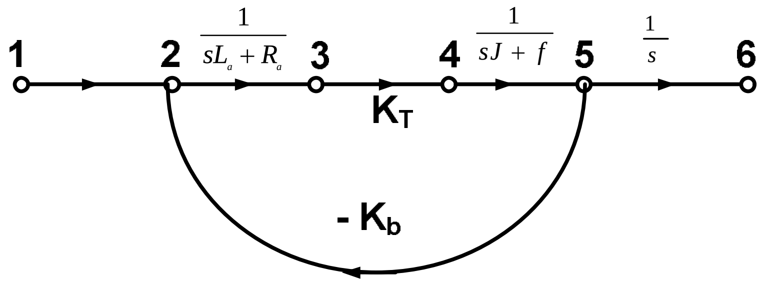
Step-2: Connect all nodes according to their signal flow.

Step-3: Each ofgains is replaced by transmittances of the branches connected between two nodes of the forward paths.

Step-4: Each ofgains is replaced by transmittances multiplied with (-1) of the branches connected between two nodes of the forward paths.



(a)



(b)

Fig.6.3. Armature type speed control of a DC motor

6.4. Mason's gain formula:

Transfer function of a system=

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_{k=1}^{N} P_k \Delta_k}{\Delta} \tag{6.1}$$

Where,

N= total number of forward paths

P_k= path gain of k^{th} forward path

Δ= 1 - (∑loop gains of all individual loops) + (∑gain product of loop gains of all possible two non-touching loops) - (∑gain product of loop gains of all possible three non-touching loops) + …

Δ_k= value of Δ after eliminating all loops that touches k^{th} forward path

Example:

Find the overall transfer function of the system given in Fig.6.1 using Mason's gain formula.

Solution:

In Fig.6.1,

No. of forward paths: $N = 2$

Path gain of forward paths: $P_1 = t_1 t_2 t_3 t_4 t_5$ and $P_2 = t_6 t_3 t_4 t_5$

Loop gain of individual loops: $L_1 = -t_2 t_3 t_6$, $L_2 = -t_5 t_7$, $L_3 = -t_1 t_2 t_3 t_4 t_5 t_8$ and $L_4 = -t_9 t_3 t_4 t_5 t_8$

No. of two non-touching loops = 2 i.e. L_1 and L_2

No. of more than two non-touching loops = 0

$$\Delta = 1 - \left(L_1 + L_2 + L_3 + L_4\right) + \left(L_1 L_2\right) - 0 = 1 - L_1 - L_2 - L_3 - L_4 + L_1 L_2$$

$$\Delta_1 = 1 - 0 = 1 \text{and} \Delta_2 = 1 - 0 = 1$$

$$G(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\Rightarrow G(s) = \frac{\left(t_1 t_2 t_3 t_4 t_5\right)(1) + \left(t_6 t_3 t_4 t_5\right)(1)}{1 + t_2 t_3 t_6 + t_5 t_7 + t_1 t_2 t_3 t_4 t_5 t_8 + t_9 t_3 t_4 t_5 t_8 + t_2 t_3 t_5 t_6 t_7}$$

$$\Rightarrow G(s) = \frac{t_1 t_2 t_3 t_4 t_5 + t_6 t_3 t_4 t_5}{1 + t_2 t_3 t_6 + t_5 t_7 + t_1 t_2 t_3 t_4 t_5 t_8 + t_9 t_3 t_4 t_5 t_8 + t_2 t_3 t_5 t_6 t_7}$$

CHAPTER#7

7. Feedback Characteristics of Control System

7.1. Feedback and Non-feedback Control systems

Non-feedback control system: It is a control system that does not have any feedback paths. It is also known as open-loop control system. It is shown in Fig.7.1 (a) and (b).

Feedback control system: It is a control system that has at least one feedback path. It is also known as closed-loop control system. It is shown in Fig.7.2 (a) and (b).

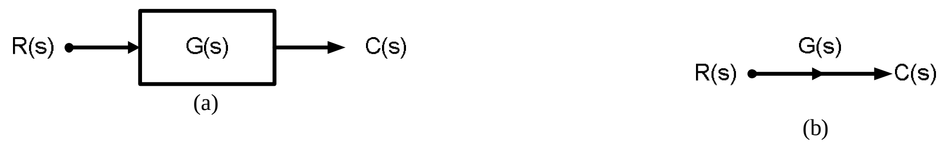


Fig.7.1. (a) Block diagram of a non-feedback control system and (b) SFG of a non-feedback control system

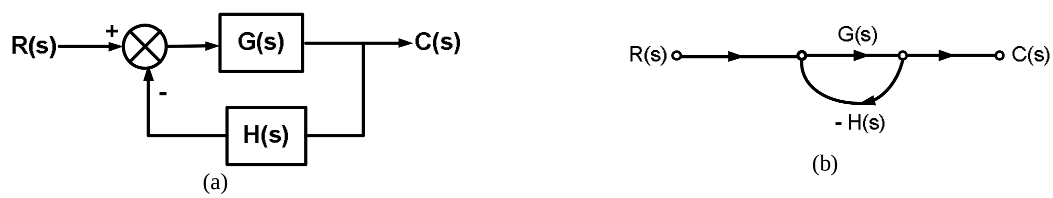


Fig.7.2. (a) Block diagram of a feedback control system and (b) SFG of a feedback control system

7.2. Types of Feedback in a Control system

7.2.1. Degenerative feedback control system: It is a control system where the feedback signal opposes the input signal. Here,

Error or actuating signal = (Input signal) – (Feedback signal).

Referring Fig.7.3,

$$E(s) = R(s) - B(s) \tag{7.1}$$

and

$$T_1(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{7.2}$$

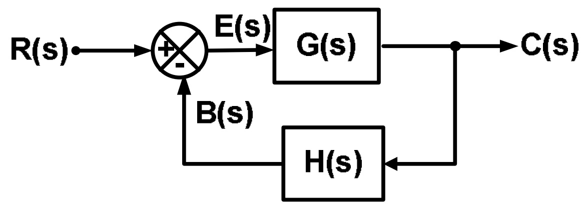


Fig.7.3. (a) Block diagram of a degenerative feedback control system

7.2.2. Regenerative feedback control system: It is a control system where the feedback signal supports or adds the input signal. Here,

Error or actuating signal = (Input signal) + (Feedback signal).

Referring Fig.7.4,

$$E(s) = R(s) + B(s) \tag{7.3}$$

and

$$T_2(s) = \frac{G(s)}{1 - G(s)H(s)} \tag{7.4}$$

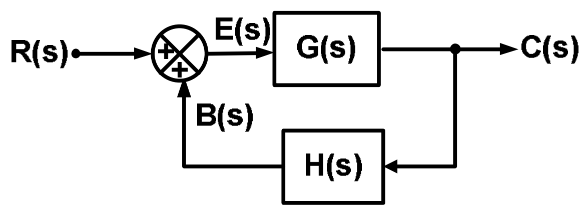


Fig.7.4. Block diagram of a regenerative feedback control system

7.3. Effect of parameter variation on overall gain of a degenerative Feedback Control system

The overall gain or transfer function of a degenerative feedback control system depends upon these parameters i.e. (i) variation in parameters of plant, and (ii) variation in parameter of feedback system and (ii) disturbance signals.

The term sensitivity is a measure of the effectiveness of feedback on reducing the influence of any of the above described parameters. For an example, it is used to describe the relative variations in the overall Transfer function of a system T(s) due to variation in G(s).

$$sensitivity = \frac{percentage\ change\ in\ T(s)}{percentage\ change\ in\ G(s)}$$

7.3.1. Effect of variation in G(s) on T(s) of a degenerative Feedback Control system

In an open-loop system,

$$C(s) = G(s)R(s)$$

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C(s) + \Delta C(s) = \left[G(s) + \Delta G(s)\right]R(s)$$
$$\Rightarrow C(s) + \Delta C(s) = G(s)R(s) + \Delta G(s)R(s)$$

$$\Rightarrow \Delta C(s) = \Delta G(s)R(s) \tag{7.5}$$

In an closed-loop system,

$$C(s)=\frac{G(s)}{1+G(s)H(s)}R(s)$$

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C(s)+\Delta C(s)=\frac{\left[G(s)+\Delta G(s)\right]}{1+\left[G(s)+\Delta G(s)\right]H(s)}R(s)$$

$$\Rightarrow C(s)+\Delta C(s)=\frac{G(s)+\Delta G(s)}{1+G(s)H(s)+\Delta G(s)H(s)}R(s)$$

Since, |G(s)| >> |ΔG(s)|, then $G(s)H(s) \gg \Delta G(s)H(s)$. Therefore, $\Delta G(s)H(s)$ is neglected. Now,

$$C(s)+\Delta C(s)=\frac{G(s)+\Delta G(s)}{1+G(s)H(s)}R(s)$$

$$\Rightarrow C(s)+\Delta C(s)=\frac{G(s)}{1+G(s)H(s)}R(s)+\frac{\Delta G(s)}{1+G(s)H(s)}R(s)$$

Or

$$\Delta C(s)=\frac{\Delta G(s)}{1+G(s)H(s)}R(s) \tag{7.6}$$

Comparing eq (42 and (43), it is clear that $\Delta C_{(open\ loop)} = (1+GH)\,\Delta C_{(closed\ loop)}$

This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in G(s) is given by

$$S_G^T=\frac{\partial T/T}{\partial G/G}=\frac{\partial T}{\partial G}\times\frac{G}{T} \tag{7.7}$$

For open-loop system,

$$S_G^T=\frac{\partial T/T}{\partial G/G}=\frac{\partial G}{\partial G}\times\frac{G}{G}=1 \tag{7.8}$$

For closed-loop system,

$$S_G^T=\frac{\partial T/T}{\partial G/G}=\frac{(1+GH)-GH}{(1+GH)^2}\times\frac{G}{G/(1+GH)}=\frac{1}{(1+GH)} \tag{7.9}$$

Therefore, it is proved that $S^T_{G\,(open\ loop)} = (1+GH) S^T_{G\,(closed\ loop)}$. Hence, the effect of parameter variation in case of closed loop system is reduced by a factor of $\frac{1}{(1+GH)}$.

7.3.2. Effect of variation in H(s) on T(s) of a degenerative Feedback Control system

This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in H(s) is given by

$$S_H^T = \frac{\partial T/T}{\partial H/H} = \frac{\partial T}{\partial H} \times \frac{H}{T} \tag{7.10}$$

For closed-loop system,

$$S_H^T = \frac{\partial T}{\partial H} \times \frac{H}{T} = G\left[\frac{-G}{\left(1+GH\right)^2}\right] \times \frac{H}{G/\left(1+GH\right)} = \frac{-GH}{\left(1+GH\right)} \tag{7.11}$$

For higher value of GH, sensitivity S_H^T approaches unity. Therefore, change in H affects directly the system output.

Equation Chapter (Next) Section 1

MODULE#2

CHAPTER#8

8. Time Domain Analysis of Control Systems

8.1. Time response

Time response c(t)is the variation of output with respect to time. The part of time response that goes to zero after large interval of time is called transient response $c_{tr}(t)$. The part of time response that remains after transient response is called steady-state response $c_{ss}(t)$.

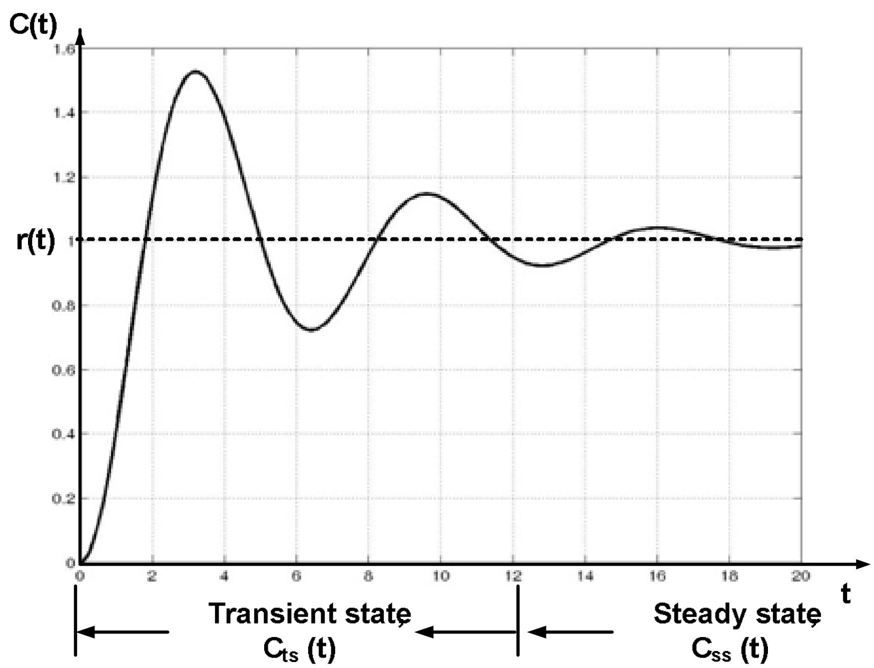


Fig.7.1. Time response of a system

8.2. System dynamics

System dynamics is the study of characteristic and behaviour of dynamic systems

i.e.

i. Differential equations: First-order systems and Second-order systems,
ii. Laplace transforms,
iii. System transfer function,
iv. Transient response: Unit impulse, Step and Ramp

Laplace transforms convert differential equations into algebraic equations. They are related to frequency response

$$\mathbf{L}\left\{\mathbf{x}(\mathbf{t})\right\} = \mathbf{X}(\mathbf{s}) = \int_0^\infty x(t)e^{-st}dt \tag{8.1}$$

No.	Function	Time-domain x(t)= $\mathcal{L}^{-1}\{X(s)\}$	Laplace domain X(s)= $\mathcal{L}\{x(t)\}$
1	Delay	$\delta(t-\tau)$	$e^{-\tau s}$
2	Unit impulse	$\delta(t)$	1
3	Unit step	u(t)	$\frac{1}{s}$
4	Ramp	t	$\frac{1}{s^2}$
5	Exponential decay	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
6	Exponential approach	$\left(1-e^{-\alpha t}\right)$	$\frac{\alpha}{s(s+\alpha)}$
7	Sine	sin ωt	$\frac{\omega}{s^2+\omega^2}$
8	Cosine	cos ωt	$\frac{s}{s^2+\omega^2}$
9	Hyperbolic sine	sinh αt	$\frac{\alpha}{s^2-\alpha^2}$
10	Hyperbolic cosine	cosh αt	$\frac{s}{s^2-\alpha^2}$
11	Exponentially decaying sine wave	$e^{-\alpha t}\sin\omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
12	Exponentially decaying cosine wave	$e^{-\alpha t}\cos\omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

8.3. Forced response

$$C(s)=G(s)R(s)=\frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}R(s) \qquad (8.2)$$

R(s) input excitation

8.4. Standard test signals

8.4.1. Impulse Signal: An impulse signal δ(t) is mathematically defined as follows.

$$\delta(t)=\left.\begin{matrix} undefined & ;t=0 \\ 0 & ;t\neq 0 \end{matrix}\right\} \qquad (8.3)$$

Laplace transform of impulse signal is

$$\delta(s) = 1 \tag{8.4}$$

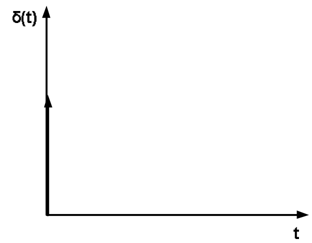
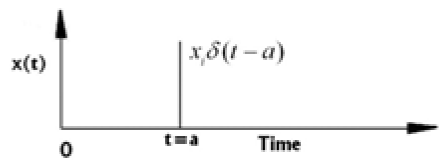


Fig.7.2. Impulse signal

Dirac delta function

$$x(t) = x_i \delta(t-a) \tag{8.5}$$



Integral property of Dirac delta function

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-t_o)dt = \phi(t_o) \tag{8.6}$$

Laplace transform of an impulse input

$$X(s) = \int_{0}^{\infty} e^{-st} x_i \delta(t-a)dt = x_i e^{-sa} \tag{8.7}$$

8.4.2. Step Signal: A step signal u(t) is mathematically defined as follows.

$$u(t) = \begin{matrix} 0 & ;t<0 \\ K & ;t \geq 0 \end{matrix}\Bigg\} \tag{8.8}$$

Laplace transform of step signal is

$$U(s) = \frac{K}{s} \tag{8.9}$$

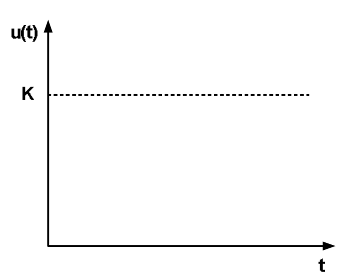


Fig.7.2. Step signal

8.4.3. Ramp Signal: A step signal r(t) is mathematically defined as follows.

$$r(t) = \left.\begin{matrix} 0 & ;t<0 \\ Kt & ;t \geq 0 \end{matrix}\right\} \quad (8.10)$$

Laplace transform of ramp signal is

$$R(s) = \frac{K}{s^2} \quad (8.11)$$

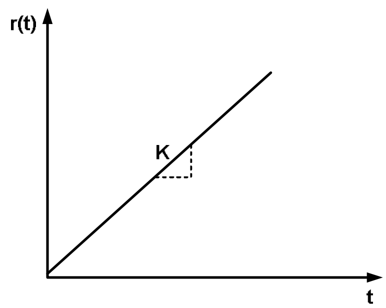


Fig.7.3. Ramp signal

8.4.4. Parabolic Signal A step signal a(t) is mathematically defined as follows.

$$a(t) = \left.\begin{matrix} 0 & ;t<0 \\ \frac{Kt^2}{2} & ;t \geq 0 \end{matrix}\right\} \quad (8.12)$$

Laplace transform of parabolic signal is

$$A(s) = \frac{K}{s^3} \quad (8.13)$$

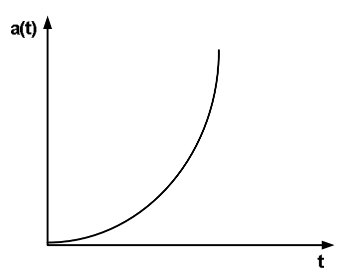


Fig.7.4. Parabolic signal

8.4.5. Sinusoidal Signal A sinusoidal x(t) is mathematically defined as follows.

$$x(t) = \sin \omega t \tag{8.14}$$

Laplace transform of sinusoidal signal is

$$X(s) = \int_0^{\infty} e^{-st} \sin \omega t \, dt = \frac{\omega}{s^2 + \omega^2} \tag{8.15}$$

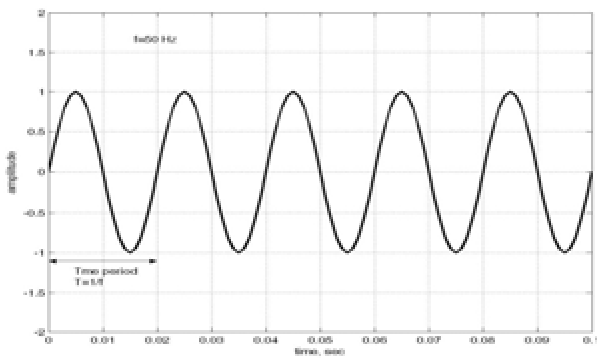


Fig.7.4. Sinusoidal signal

8.5. Steady-state error:

A simple closed-loop control system with negative feedback is shown as follows.

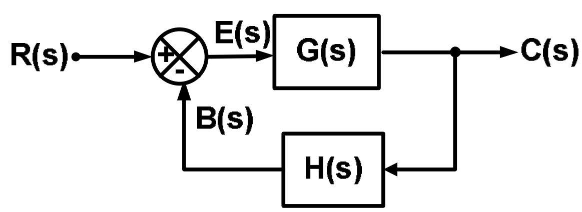


Fig.7.5. A simple closed-loop control system with negative feedback

Here,

$$E(s) = R(s) - B(s) \tag{8.16}$$

$$B(s) = C(s)H(s) \tag{8.17}$$

$$C(s) = E(s)G(s) \tag{8.18}$$

Applying (1) in (9),

$$E(s) = R(s) - C(s)H(s) \tag{8.19}$$

Using (11) in (12),

$$E(s) = R(s) - E(s)G(s)H(s) \tag{8.20}$$

$$\Rightarrow \left[1 + G(s)H(s)\right]E(s) = R(s) \tag{8.21}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)H(s)} \tag{8.22}$$

Steady-state error,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \tag{8.23}$$

Using (15) in (16),

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} \tag{8.24}$$

Therefore, steady-state error depends on two factors, i.e.

(a) type and magnitude of R(s)
(b) open-loop transfer function G(s)H(s)

8.6. Types of input and Steady-state error:

8.6.1. Step Input

$$R(s) = \frac{A}{s} \tag{8.25}$$

Using (18) in (17),

$$e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{A}{s}\right)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{A}{1+G(s)H(s)} \tag{8.26}$$

$$\Rightarrow e_{ss} = \frac{A}{1+\lim_{s\to 0} G(s)H(s)} = \frac{A}{1+K_P} \tag{8.27}$$

Where,

$$K_P = \lim_{s\to 0} G(s)H(s) \tag{8.28}$$

8.6.2. Ramp Input

$$R(s) = \frac{A}{s^2} \tag{8.29}$$

Using (18) in (17),

$$e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{A}{s^2}\right)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{A}{s\left[1+G(s)H(s)\right]}$$

$$\Rightarrow e_{ss} = \lim_{s\to 0} \frac{A}{s+sG(s)H(s)} \tag{8.30}$$

$$\Rightarrow e_{ss} = \frac{A}{\lim_{s\to 0} sG(s)H(s)} = \frac{A}{K_V}$$

Where,

$$K_V = \lim_{s \to 0} sG(s)H(s) \tag{8.31}$$

8.6.3. Parabolic Input

$$R(s) = \frac{A}{s^3} \tag{8.32}$$

Using (18) in (17),

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} \frac{s\left(\frac{A}{s^3}\right)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{A}{s^2\left[1+G(s)H(s)\right]} \\ \Rightarrow e_{ss} &= \lim_{s \to 0} \frac{A}{s^2 + s^2G(s)H(s)} \\ \Rightarrow e_{ss} &= \frac{A}{\lim_{s \to 0} s^2G(s)H(s)} = \frac{A}{K_A} \end{aligned} \tag{8.33}$$

Where,

$$K_A = \lim_{s \to 0} s^2G(s)H(s) \tag{8.34}$$

Types of input and steady-state error are summarized as follows.

Error Constant	**Equation**	**Steady-state error (e_{ss})**
Position Error Constant (K_P)	$K_P = \lim_{s \to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1+K_P}$
Velocity Error Constant (K_V)	$K_V = \lim_{s \to 0} sG(s)H(s)$	$e_{ss} = \frac{A}{K_V}$
Acceleration Error Constant (K_A)	$K_A = \lim_{s \to 0} s^2G(s)H(s)$	$e_{ss} = \frac{A}{K_A}$

8.7. Types of open-loop transfer function G(s)H(s)and Steady-state error:

8.7.1. Static Error coefficient Method

The general form of G(s)H(s) is

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{s^j(1+T_as)(1+T_bs)...(1+T_ms)} \tag{8.35}$$

Here, j = no. of poles at origin (s = 0)

or, type of the system given by eq (28) is j.

8.7.1.1. Type 0

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)...(1+T_ns)}{(1+T_as)(1+T_bs)...(1+T_ms)} \tag{8.36}$$

Here,

$$K_P = \lim_{s \to 0} G(s)H(s) = K \tag{8.37}$$

Therefore,

$$e_{ss} = \frac{A}{1+K} \tag{8.38}$$

8.7.1.2. Type 1

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)...(1+T_n s)}{s(1+T_a s)(1+T_b s)...(1+T_m s)} \tag{8.39}$$

Here,

$$K_V = \lim_{s \to 0} sG(s)H(s) = K \tag{8.40}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{8.41}$$

8.7.1.3. Type 2

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)...(1+T_n s)}{s^2(1+T_a s)(1+T_b s)...(1+T_m s)} \tag{8.42}$$

Here,

$$K_A = \lim_{s \to 0} s^2 G(s)H(s) = K \tag{8.43}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{8.44}$$

Steady-state error and error constant for different types of input are summarized as follows.

Type	**Step input**		**Ramp input**		**Parabolic input**	
	K_P	e_{ss}	K_V	e_{ss}	K_A	e_{ss}
Type 0	K	$\frac{A}{1+K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{A}{K}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{A}{K}$

The static error coefficient method has following advantages:

- Can provide time variation of error
- Simple calculation

But, the static error coefficient method has following demerits:

- Applicable only to stable system
- Applicable only to three standard input signals
- Cannot give exact value of error. It gives only mathematical value i.e. 0 or ∞

8.7.2. Generalized Error coefficient Method

From eq (15),

$$E(s)=\left[\frac{1}{1+G(s)H(s)}\right]R(s)$$

So,

$$E(s)=F_1(s)F_2(s) \tag{8.45}$$

Where, $F_1=\dfrac{1}{1+G(s)H(s)}$ and $F_2(s)=R(s)$

Using convolution integral to eq (38)

$$e(t)=\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau=\int_0^t f_1(\tau)r(t-\tau)\,d\tau \tag{8.46}$$

Using Taylor's series of expansion to $r(t-\tau)$,

$$r(t-\tau)=r(t)-\tau r'(t)+\frac{\tau^2}{2!}r''(t)-\frac{\tau^3}{3!}r'''(t)+\ldots \tag{8.47}$$

Now, applying eq (40) in eq (39),

$$e(t)=\int_0^t f_1(\tau)r(t)\,d\tau-\int_0^t \tau r'(t)\,f_1(\tau)d\tau+\int_0^t \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^t \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots \tag{8.48}$$

Now, steady-state error, e_{ss} is

$$e_{ss}=\lim_{t\to\infty}e(t) \tag{8.49}$$

Therefore,

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{t\to\infty}\left[\int_0^t f_1(\tau)r(t)\,d\tau-\int_0^t \tau r'(t)\,f_1(\tau)d\tau+\int_0^t \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^t \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots\right]$$

$$\Rightarrow e_{ss}=\int_0^\infty f_1(\tau)r(t)\,d\tau-\int_0^\infty \tau r'(t)\,f_1(\tau)d\tau+\int_0^\infty \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^\infty \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots \tag{8.50}$$

Eq (44) can be rewritten as

$$e_{ss}=C_0r(t)+C_1r'(t)+\frac{C_2}{2!}r''(t)+\frac{C_3}{3!}r'''(t)+\ldots \tag{8.51}$$

Where, C_0, C_1, C_2, C_3, etc. are dynamic error coefficients. These are given as

$$
\begin{aligned}
C_0 &= \int_0^\infty f_1(\tau)d\tau = \lim_{s\to 0} F_1(s) \\
C_1 &= \int_0^\infty -\tau f_1(\tau)d\tau = \lim_{s\to 0} \frac{dF_1(s)}{ds} \\
C_2 &= \int_0^\infty \frac{\tau^2}{2!} f_1(\tau)d\tau = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} \\
C_3 &= \int_0^\infty -\frac{\tau^3}{3!} f_1(\tau)d\tau + = \lim_{s\to 0} \frac{d^3F_1(s)}{ds^3}
\end{aligned}
\quad \text{, and so on…} \tag{8.52}
$$

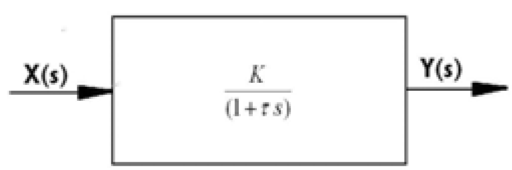
8.8. First-order system:

A Governing differential equation is given by

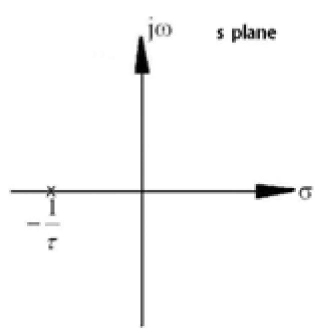
$$y + \tau \dot{y} = Kx(t) \tag{8.53}$$

Where, Time constant, sec = τ ,
Static sensitivity (units depend on the input and output variables) = K ,
y(t) is response of the system and
x(t) is input excitation
The System transfer function is

$$\frac{Y(s)}{X(s)} = G(s) = \frac{K}{(1+\tau s)} \tag{8.54}$$



Pole-zero map of a first-order system



Normalized response

In this type of response

- Static components are taken out leaving only the dynamic component
- The dynamic components converge to the same value for different physical systems of the same type or order
- Helps in recognizing typical factors of a system

8.8.1. Impulse input to a first-order system

Governing differential equation

$$y + \tau \dot{y} = Kx_i \delta(t) \tag{8.55}$$

Laplacian of the response

$$Y(s) = \frac{Kx_i}{(1+\tau s)} = \frac{Kx_i}{\tau}\left(\frac{1}{s+\frac{1}{\tau}}\right) \tag{8.56}$$

Time-domain response

$$y(t) = \frac{Kx_i}{\tau} e^{-\frac{t}{\tau}} \tag{8.57}$$

Impulse response function of a first-order system

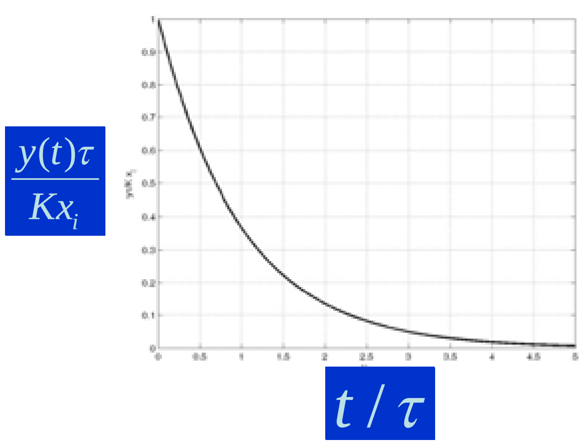
$$h(t) = \frac{K}{\tau} e^{-\frac{t}{\tau}} \tag{8.58}$$

By putting x_i=1 in the response

Response of a first-order system to any force excitation

$$y(t) = \frac{K}{\tau}\int_0^t e^{-\frac{t}{\tau}} F(t-\eta) d\eta \tag{8.59}$$

The above equation is called Duhamel's integral. Normalized response of a first-order system to impulse input is shown below.



8.8.2. Step input to a first-order system

Governing differential equation

$$y + \tau \dot{y} = Kx_i u(t) \tag{8.60}$$

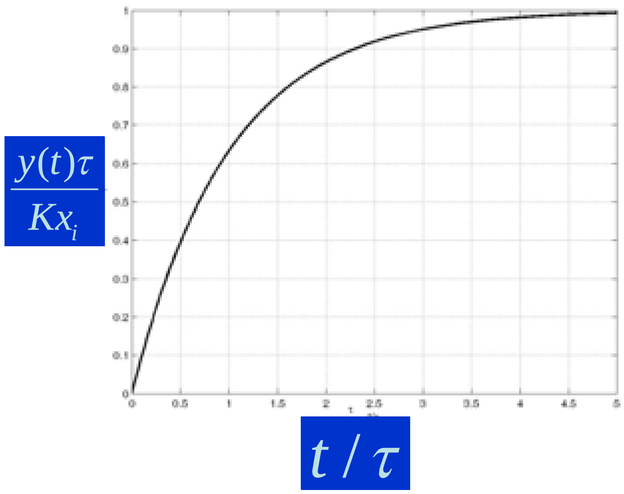
Laplacian of the response

$$Y(s) = \frac{Kx_i}{s(1+\tau s)} = \frac{Kx_i}{s} - \frac{Kx_i}{s + \frac{1}{\tau}} \tag{8.61}$$

Time-domain response

$$y(t) = Kx_i\left(1 - e^{-\frac{t}{\tau}}\right) \tag{8.62}$$

Normalized response of a first-order system to impulse input is shown below.



8.8.3. Ramp input to a first-order system

Governing differential equation

$$y + \tau\,\dot{y} = Kt \tag{8.63}$$

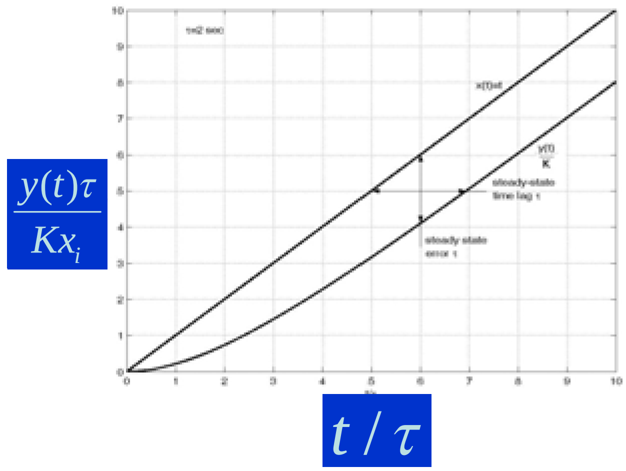
Laplacian of the response

$$Y(s) = \frac{K}{s^2(1+\tau s)} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s + \frac{1}{\tau}} \tag{8.64}$$

Time-domain response

$$\frac{y(t)}{K} = t - \tau + \tau e^{-\frac{t}{\tau}} \tag{8.65}$$

Normalized response of a first-order system to impulse input is shown below.



8.8.4. Sinusoidal input to a first-order system
Governing differential equation

$$y + \tau\,\dot{y} = KA\sin\omega t \tag{8.66}$$

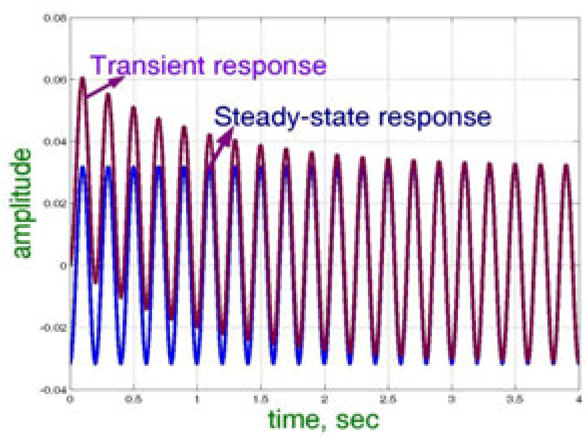
Laplacian of the response

$$Y(s) = \frac{K}{(1+\tau s)}\left(\frac{A\omega}{s^2+\omega^2}\right) = \frac{\omega}{1+(\omega\tau)^2}\left\{\frac{\tau}{s+1/\tau} - \frac{\tau s}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}\right\} \tag{8.67}$$

Time-domain response

$$\frac{y(t)}{KA} = \frac{\omega}{1+(\omega\tau)^2}\left\{\tau e^{-t/\tau} - \tau\cos\omega t + \frac{1}{\omega}\sin\omega t\right\} \tag{8.68}$$

Normalized response of a first-order system to impulse input is shown below.



8.9. Second-order system

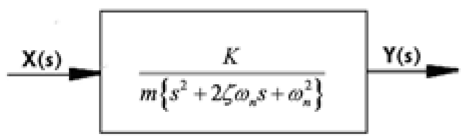
A Governing differential equation is given by

$$m\ddot{y} + c\dot{y} + ky = Kx(t) \tag{8.69}$$

Where, τ = Time constant, sec,
K = Static sensitivity (units depend on the input and output variables),
m = Mass (kg),
c = Damping coefficient (N-s/m),
k = Stiffness (N/m),
y(t) is response of the system and
x(t) is input excitation

The System transfer function is

$$\frac{Y(s)}{X(s)} = \frac{K}{m\left\{s^2 + 2\zeta\omega_n s + \omega_n^2\right\}} \tag{8.70}$$

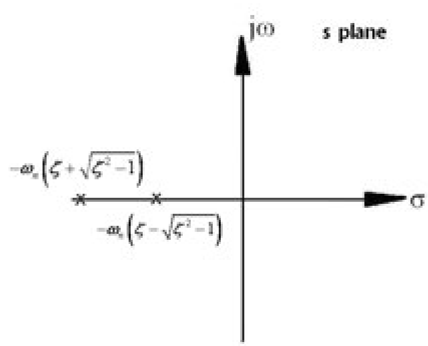


Pole-zero map

(a) ζ>1 over damped
Poles are:

$$s_{1,2} = -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) \tag{8.71}$$

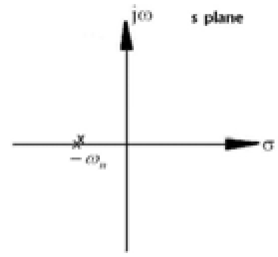
Graphically, the poles of an over damped system is shown as follows.



(b) ζ =1 critically damped
Poles are:

$$s_{1,2} = -\omega_n \tag{8.72}$$

Graphically, the poles of an critically damped system is shown as follows.



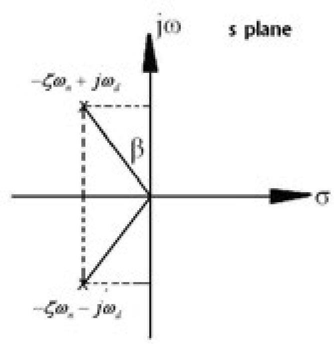
(c) ζ<1 under damped
Poles are:

$$s_{1,2} = -\omega_n\left(\zeta \pm j\sqrt{1-\zeta^2}\right)$$
$$\Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_d \tag{8.73}$$

Where, $\omega_d =$ Damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{8.74}$$

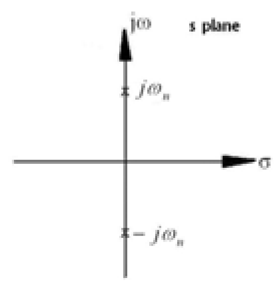
Graphically, the poles of an critically damped system is shown as follows.



Here, $\tan\beta = \dfrac{\zeta}{\sqrt{1-\zeta^2}}$

(d) ζ = 0 un-damped
Poles are:

$$s_{1,2} = -\pm j\omega_n \tag{8.75}$$



Solved problems:

1. A single degree of freedom spring-mass-damper system has the following data: spring stiffness 20 kN/m; mass 0.05 kg; damping coefficient 20 N-s/m. Determine
   (a) undamped natural frequency in rad/s and Hz
   (b) damping factor
   (c) damped natural frequency n rad/s and Hz.

If the above system is given an initial displacement of 0.1 m, trace the phasor of the system for three cycles of free vibration.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20\times 10^3}{0.05}} = 632.46 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{632.46}{2\pi} = 100.66\,\text{Hz}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{20\times 10^3 \times 0.05}} = 0.32$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 632.46\sqrt{1-0.32^2} = 600\,\text{rad/s}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{600}{2\pi} = 95.37\,\text{Hz}$$

$$y(t) = Ae^{-\zeta\omega_n t} = 0.1e^{-0.32\times 632.46t}$$

2. A second-order system has a damping factor of 0.3 (underdamped system) and an un-damped natural frequency of 10 rad/s. Keeping the damping factor the same, if the un-damped natural frequency is changed to 20 rad/s, locate the new poles of the system? What can you say about the response of the new system?

Solution:

Given, $\omega_{n1} = 10$ rad/s and $\omega_{n2} = 20$ rad/s

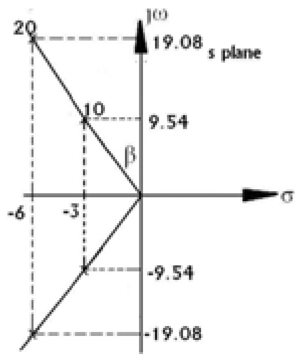
$$\omega_{d_1} = \omega_{n_1}\sqrt{1-\zeta^2} = 10\sqrt{1-0.3^2} = 9.54\,\text{rad/s}$$

$$\omega_{d_2} = \omega_{n_2}\sqrt{1-\zeta^2} = 20\sqrt{1-0.3^2} = 19.08\,\text{rad/s}$$

$$p_{1,2} = -\zeta\omega_{n_1} \pm j\omega_{d_1} = -3 \pm j9.54$$

$$p_{3,4} = -\zeta\omega_{n_2} \pm j\omega_{d_2} = -6 \pm j19.08$$

$$\tan\beta = \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{0.3}{\sqrt{1-0.3^2}} = 17.45^o$$



8.9.1. Second-order Time Response Specifications with Impulse input

(a) Over damped case (ζ>1)
General equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.76}$$

Laplacian of the output

$$\begin{aligned} Y(s) &= \frac{Kx_i}{m}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \\ &= \frac{Kx_i}{2m\omega_n\sqrt{\zeta^2-1}}\left\{\frac{1}{(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})} - \frac{1}{(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}\right\} \end{aligned} \tag{8.77}$$

Time-domain response

$$y(t) = \left[\frac{Kx_i}{m\omega_n\sqrt{\zeta^2-1}}\right] e^{-\zeta\omega_n t}\sinh\left(\omega_n\sqrt{\zeta^2-1}\right)t \tag{8.78}$$

(b) Critically damped case (ζ=1)
General equation

$$\ddot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.79}$$

Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left(\frac{1}{s^2+\omega_n^2}\right) \tag{8.80}$$

Time-domain response

$$y(t) = \left\{\frac{Kx_i}{m\omega_n}\right\}\omega_n t e^{-\omega_n t} \tag{8.81}$$

(c) Under damped case (ζ<1)

Poles are: $s_{1,2} = -\zeta\omega_n \pm j\omega_d$

General equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.82}$$

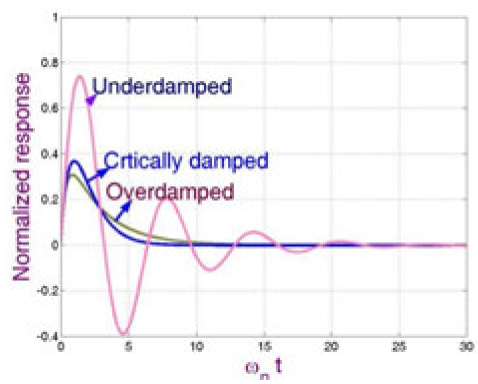
Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{(s+\zeta\omega_n + j\omega_d)(s+\zeta\omega_n - j\omega_d)}\right\} \tag{8.83}$$

Time-domain response

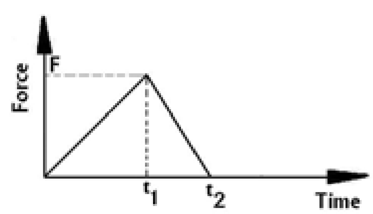
$$y(t) = \left\{\frac{Kx_i}{m\omega_d}\right\} e^{-\zeta\omega_n t}\sin\omega_d t \tag{8.84}$$

Normalized impulse-response of a second-order system with different damping factors are shown graphically as follows.



Solved problems:

3. A second-order system has an un-damped natural frequency of 100 rad/s and a damping factor of 0.3. The value of the coefficient of the second time derivative (that is m) is 5. If the static sensitivity is 10, write down the response (do not solve) for a force excitation shown in the figure in terms of the Duhamel's integral for the following periods of time: 0<t<t1, t1<t<t2 and t>t2.



Solution:

Given, Undamped natural frequency ω_n=100 rad/s

Damping factor ξ =0.3

Coefficient of the second time derivative m=5

Static sensitivity K=10

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-0.3^2} = 95.39 \text{ rad/s}$$

Here,

$$F(t) = F\frac{t}{t_1} \quad ;0 \le t < t_1$$

$$F(t) = \frac{F}{t_2 - t_1}\left(t_2 - t\right) \quad ;t_1 \le t < t_2$$

$$y(t) = \frac{K}{m\omega_d}\int_0^t F(t-\eta)e^{-\zeta\omega_n\eta}\sin\left(\omega_d\eta\right)d\eta$$

$$\Rightarrow y(t) = \frac{10F}{5\times 95.39t_1}\int_0^t e^{-0.3\times 100\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$= \frac{0.057F}{t_1}\int_0^t e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta \qquad ;0 < t < t_1,$$

$$\Rightarrow y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$+\frac{0.057F}{t_2 - t_1}\int_{t_1}^{t} e^{-30\eta}\sin\left(95.39\eta\right)(t_2 - t-\eta)d\eta \qquad ;t_1 < t < t_2 \text{ and}$$

$$\Rightarrow y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$+\frac{0.057F}{t_2 - t_1}\int_{t_1}^{t_2} e^{-30\eta}\sin\left(95.39\eta\right)(t_2 - t-\eta)d\eta \qquad ;t > t_2$$

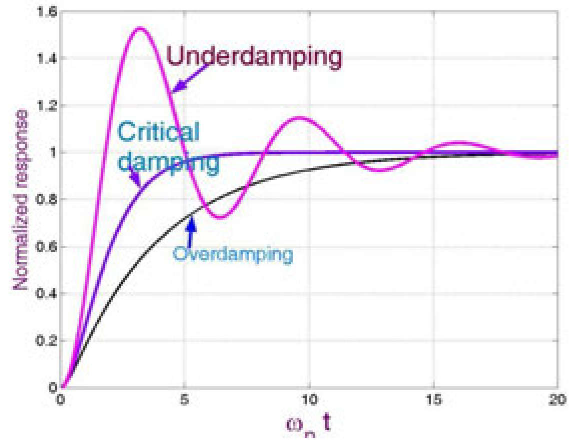
8.9.2. Second-order Time Response Specifications with step input

$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{s(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}\right\} \tag{8.85}$$

$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1-e^{-\zeta\omega_n t}\left[\cosh\left(\omega_n\sqrt{\zeta^2-1}\right)t+\frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_n\sqrt{\zeta^2-1}\right)t\right]\right\} \tag{8.86}$$

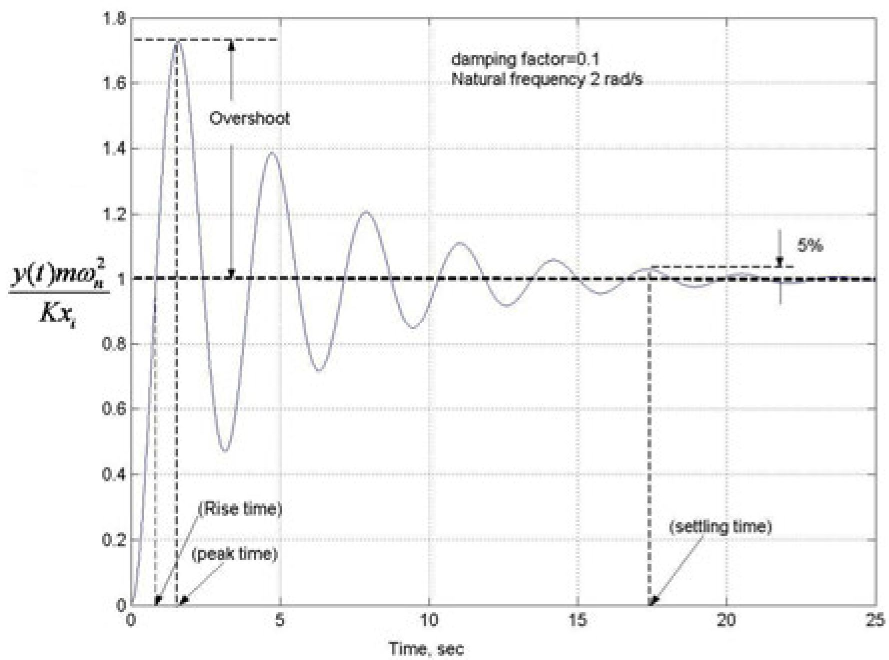
$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{s(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}\right\} \tag{8.87}$$

$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1-e^{-\zeta\omega_n t}\left[\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\, t\right]\right\} \tag{8.88}$$



8.10. Time Response Specifications with step-input for under-damped case

For under-damped case, the step-response of a second-order is shown as follows



$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \varphi)\right\} \tag{8.89}$$

$$\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{8.90}$$

For this case, different time-domain specifications are described below.

(i) Delay time, t_d

(ii) Rise time, t_r
(iii) Peak time, t_p
(iv) Peak overshoot, M_p
(v) Settling time

For unity step input,
(i)Delay time, t_d: It is the time required to reach 50% of output.

$$y(t_d) = \frac{1}{2} = 1 - \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_d + \varphi)$$

$$\Rightarrow t_d = \frac{1+0.7\zeta}{w_n} \tag{8.91}$$

(ii) Rise time, t_r: The time required by the system response to reach from 10% to 90% of the final value for over-damped case, from 0% to 100% of the final value for under-damped case and from 5% to 95% of the critically value for over-damped case.
.

$$y(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi)$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi) = 0$$

$$\Rightarrow \omega_d t_r + \varphi = \pi$$

$$\Rightarrow t_r = \frac{\pi - \varphi}{w_d} \tag{8.92}$$

(iii) Peak time, t_p: The time required by the system response to reach the first maximum value.

$$\frac{dy(t_p)}{dt} = 0$$

$$\Rightarrow \frac{d\left[1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p + \varphi)\right]}{dt} = 0$$

$$\Rightarrow \frac{d\left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p + \varphi)\right]}{dt} = 0$$

$\Rightarrow w_d t_p + \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = n\pi + \varphi$; where $n = 1, 2, 3, \ldots$

For n=1,

$$\Rightarrow w_d t_p = n\pi$$

$$\Rightarrow t_p = \frac{n\pi}{w_d} \tag{8.93}$$

(iv) Peak overshoot, M_p: It is the time required to reach 50% of output.

$$M_p(\%) = 100 \times \frac{y(t_p) - 1}{1}$$

$$\Rightarrow M_p(\%) = 100 \times \left[1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \varphi) - 1 \right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \varphi) \right] = 100 \times \left[-\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \varphi) \right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[-\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\omega_d \frac{\pi\zeta}{\sqrt{1-\zeta^2}} + \varphi) \right] = 100 \times \left[-\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \varphi) \right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\varphi \right] = 100 \times \left[\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} \right]$$

$$\Rightarrow M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{8.94}$$

(iv) Settling time, t_s: It is the time taken by the system response to settle down and stay with in $\pm 2\%$ or $\pm 5\%$ its final value.
For $\pm 2\%$ error band,

$$t_s = \frac{4}{\zeta w_n} \tag{8.95}$$

For $\pm 5\%$ error band,

$$t_s = \frac{3}{\zeta w_n} \tag{8.96}$$

Sl. No.	**Time Specifications**	
	Type	**Formula**
1	Delay time	$t_d = \frac{1+0.7\zeta}{w_n}$
2	Rise time	$t_r = \frac{\pi - \varphi}{w_d}$
3	Peak time	$t_p = \frac{\pi}{w_d}$
4	Maximum overshoot	$M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
5	Settling time	$t_s = \frac{4}{\zeta w_n}$

Solved Problems:

1. Consider the system shown in Figure 1. To improve the performance of the system a feedback is added to this system, which results in Figure 2. Determine the value of K so that the damping ratio of the new system is 0.4. Compare the overshoot, rise time, peak time and settling time and the nominal value of the systems shown in Figures 1 and 2.

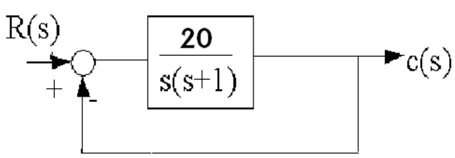


Figure 1

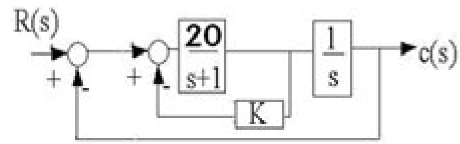


Figure 2

Solution:

For Figure 1,

$$\frac{c(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{20}{s(s+1)}}{1+\frac{20}{s(s+1)}} = \frac{20}{s^2+s+20}$$

Here, $\omega_n^2 = 20$ and $2\zeta\omega_n = 1$

$$\omega_n = \sqrt{20} \text{ rad/s and } \zeta = \frac{1}{2\omega_n} = \frac{1}{2\times\sqrt{20}} = 0.112$$

For Figure 2,

$$\frac{c(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{20}{s(s+1+20K)}}{1+\frac{20}{s(s+1+20K)}} = \frac{20}{s^2+(1+20K)s+20}$$

Here, $\omega_n^2 = 20$ and $2\zeta\omega_n = 1+20\text{K}$

$$\omega_n = \sqrt{20} \text{ rad/s}$$

But, given that $\zeta = \frac{1+20K}{2\omega_n} = \frac{1+20K}{2\sqrt{20}} = 0.4$

$\Rightarrow K = 0.128$

Transient characteristics of Figures 1 and 2

CharacteristicS	**Figure 1**	**Figure 2**
Overshoot, M_p	70%	25%
Rise time, $t_{r,}$ sec	0.38	0.48
Peak time, t_p, sec	0.71	0.77
Settling time (2%), sec	8	2.24
Steady-state value, c_∞	1.0	1.0

Equation Chapter (Next) Section 1

1.1. Transient Response using MATLAB

Program 1: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{3s+20}{s^2+5s+36}$

Solution:

```
>> num=[3 20]
num=
3          20
>> den=[1 5 36]
den=
1          5          36
>>sys=tf(num,den)
Transfer function:
      3s+20
--------------------
    s^2+5s+36
>>step(sys)
```

Program 2: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{20}{s^2+4s+25}$

Solution:

```
>> num=[20]
num=
           20
>> den=[1 425]
den=
           1          4          25
>>sys=tf(num,den)
Transfer function:
20
--------------------
    s^2+4s+25
>>step(sys)
```

2. Stability

2.1. Concept of stability

Stability is a very important characteristic of the transient performance of a system. Any working system is designed considering its stability. Therefore, all instruments are stable with in a boundary of parameter variations.

A linear time invariant (LTI) system is stable if the following two conditions are satisfied.

(i) **Notion-1:** When the system is excited by a bounded input, output is also bounded.

Proof:

A SISO system is given by

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} \tag{9.1}$$

So,

$$c(t) = \alpha^{-1}\left[G(s)R(s)\right] \tag{9.2}$$

Using convolution integral method

$$c(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau \tag{9.3}$$

$g(\tau) = \alpha^{-1}G(s) =$ impulse response of the system

Taking absolute value in both sides,

$$\left|c(t)\right| = \left|\int_0^\infty g(\tau) r(t-\tau) d\tau\right| \tag{9.4}$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand

$$\begin{aligned} &\left|c(t)\right| \le \int_0^\infty \left|g(\tau) r(t-\tau) d\tau\right| \\ &\Rightarrow \left|c(t)\right| \le \int_0^\infty \left|g(\tau) r(t-\tau)\right| d\tau \\ &\Rightarrow \left|c(t)\right| \le \int_0^\infty \left|g(\tau)\right| \left|r(t-\tau)\right| d\tau \end{aligned} \tag{9.5}$$

Let, r(t) and c(t) are bounded as follows.

$$\begin{aligned} \left|r(t)\right| &\le M_1 < \infty \\ \left|c(t)\right| &\le M_2 < \infty \end{aligned} \tag{9.6}$$

Then,

$$|c(t)| \le M_1 \int_0^\infty |g(\tau)| d\tau \le M_2 \tag{9.7}$$

Hence, first notion of stability is satisfied if $\int_0^\infty |g(\tau)| d\tau$ is finite or integrable.

(ii) **Notion-2:** In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

2.2. Effect of location of poles on stability

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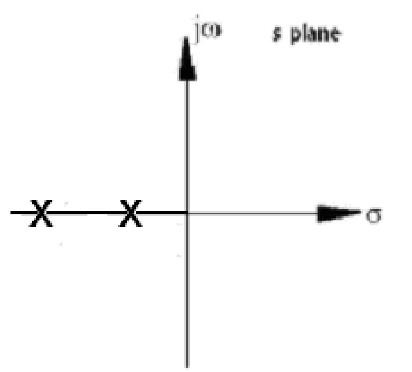
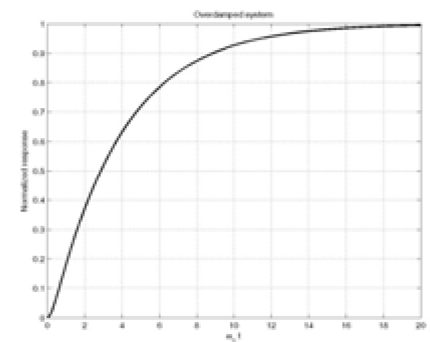
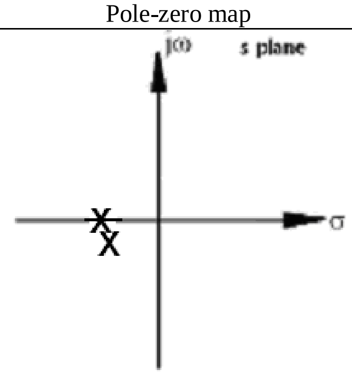
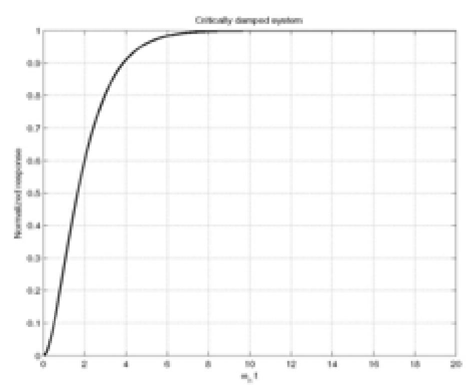
<tr><th>Pole-zero map</th><th>Normalized response</th></tr>
<tr><td colspan="2">Over-damped close-loop poles</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Critically damped close-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
<tr><td>
</td><td>
</td></tr>
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</table>

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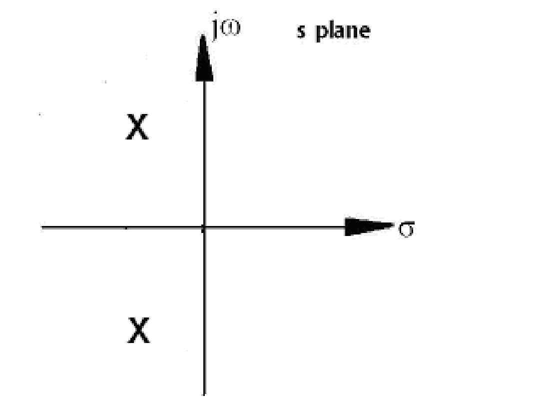
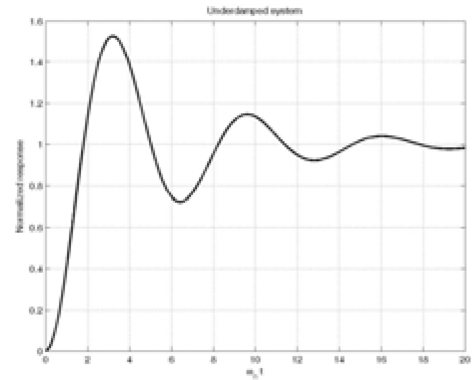
<tr><td>
</td><td></td></tr>
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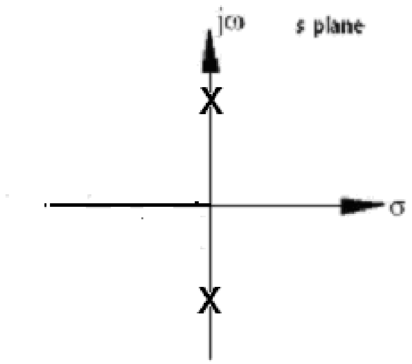
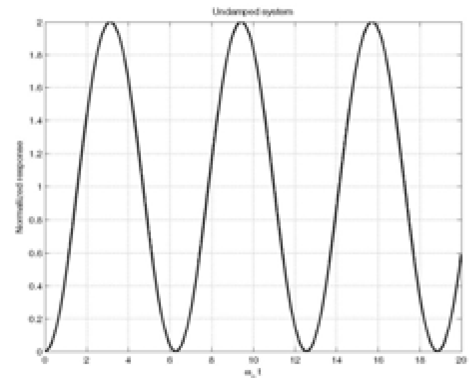
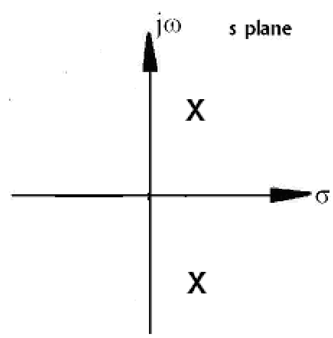
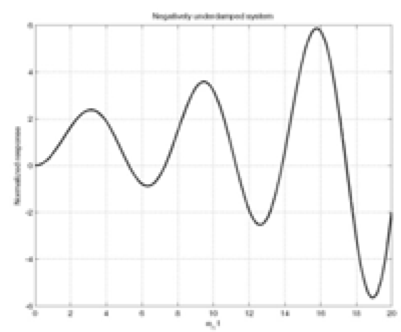
<tr><td>
</td><td></td></tr>
<tr><td colspan="2">Negative Under-dampedclose-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
<tr><td>
</td><td></td></tr>
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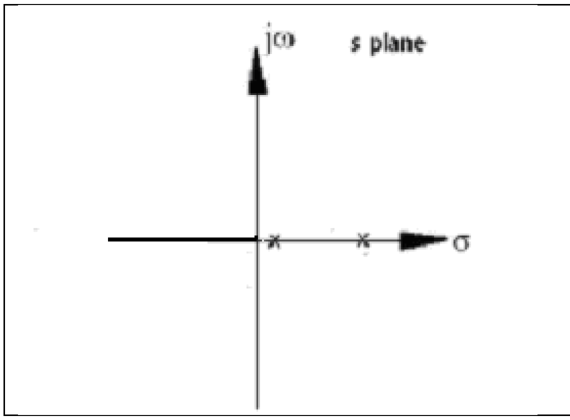
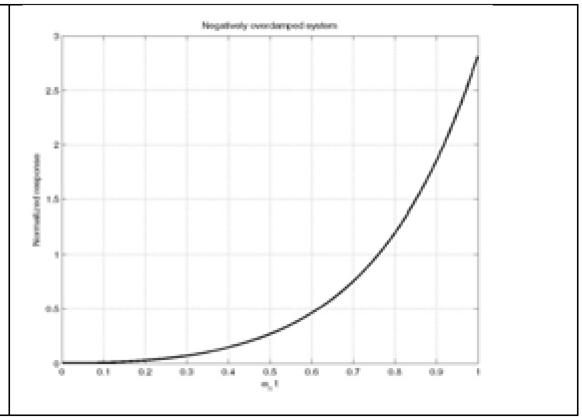




2.3. Closed-loop poles on the imaginary axis

Closed-loop can be located by replace the denominator of the close-loop response with s=jω.

Example:

1. Determine the close-loop poles on the imaginary axis of a system given below.

$$G(s) = \frac{K}{s(s+1)}$$

Solution:

Characteristics equation, $B(s) = s^2 + s + K = 0$

Replacing $s = jw$

$$B(j\omega) = (j\omega)^2 + (j\omega) + K = 0$$

$$\Rightarrow (K - \omega^2) + j\omega = 0$$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \sqrt{K}$ and $\omega = 0$

Therefore, Closed-loop poles do not cross the imaginary axis.

2. Determinetheclose the imaginary axis of a system given below.

$B(s) = s^3 + 6s^2 + 8s + K = 0$.

Solution:

Characteristics equation,

$$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = 0$$

$$\Rightarrow (K - 6\omega^2) + j(8\omega - \omega^3) = 0$$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \pm\sqrt{8}$ rad/s and $K = 6\omega^2 = 48$

Therefore, Close-loop poles cross the imaginary axis for K>48.

2.4. Routh-Hurwitz's Stability Criterion

General form of characteristics equation,

$$B(s)=a_n s^n + a_{n-1}s^{n-1}+\cdots a_1 s + a_0 = 0$$

$$\Rightarrow (s-r_1)(s-r_2)\cdot\cdot(s-r_n)=0$$

Where, r_i = Roots of the characteristics equation

2.4.1. Necessary condition of stability:

Coefficients of the characteristic polynomial must be positive.

Example:

3. Consider a third order polynomial $B(s) = s^3 + 3s^2 + 16s + 130$. Although the coefficients of the above polynomial are positive, determine the roots and hence prove that the rule about coefficients being positive is only a necessary condition for the roots to be in the left s-plane.

Solution:

Characteristics equation, $B(s) = s^3 + 3s^2 + 16s + 130 = 0$

By using Newton-Raphson's method $r_1 = -5$ and $r_{2,3} = 1 \pm j5$

Therefore, from the above example, the condition that coefficients of a polynomial should be positive for all its roots to be in the left s-plane is only a necessary condition.

2.4.2. Sufficient condition of stability:

2.4.2.1. Method I (using determinants)

The coefficients of the characteristics equation are represented by determinant form as follows.

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ & & & \end{vmatrix} \quad (9.8)$$

Here, the determinant decreases by two along the row by one down the column. For stability, the following conditions must satisfy.

$$\Delta_1 = a_{n-1} > 0, \Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} > 0 \cdots \quad (9.9)$$

2.4.2.2. Method II (using arrays)

The coefficients of the characteristics equation are represented by array form as follows.

$$\begin{array}{c|ccc} s^n & a_n & a_{n-2} & a_{n-4} \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\ \vdots & & & \end{array} \qquad (9.10)$$

Where,

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}}$$
$$b_{n-3} = \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}} \qquad (9.11)$$
$$c_{n-1} = \frac{(b_{n-1})(a_{n-3}) - a_{n-1}(b_{n-3})}{b_{n-1}}$$

For stability, the following conditions must satisfy.
The number of roots of B(s) with positive real parts is equal to the number of sign changes a_n, a_{n-1}, b_{n-1}, c_{n-1}, etc.

Example:

4. Find stability of the following system given by $G(s) = \dfrac{K}{s(s+1)}$ and $H(s) = 1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $T(s) = \dfrac{G(s)}{1 + G(s)H(s)} = \dfrac{\dfrac{K}{s(s+1)}}{1 + \dfrac{K}{s(s+1)}} = \dfrac{K}{s^2 + s + K}$

Method-I,

Characteristics equation, $B(s) = s^2 + s + K = 0$

Here, $\Delta_1 = 1$

$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & K \end{vmatrix} = K$

For stability, $\Delta_1 > 0$

$\Delta_2 > 0$

The system is always stable for K>0.

Method-II,

Characteristics equation, $B(s) = s^2 + s + K = 0$

Here, Routh array is

$$\begin{array}{c|cc} s^2 & 1 & K \\ s^1 & 1 & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array.Therefore, the system is always stable for K>0.

5. Find stability of the following system given by $G(s) = \dfrac{K}{s(s+2)(s+4)}$ and $H(s) = 1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}} = \dfrac{K}{s^3+6s^2+8s+K}$

Method-I,

General form of characteristics equation, $B(s) = a_3s^3 + a_2s^2 + a_1s + a_0 = 0$

And in this system, characteristics equation is $B(s) = s^3 + 6s^2 + 8s + K = 0$

Here,sufficient condition of stability suggests

$$\Delta_1 = 8 > 0, \Delta_2 = \begin{vmatrix} 6 & K \\ 1 & 8 \end{vmatrix} = (48 - K) > 0,$$

$$\Delta_3 = \begin{vmatrix} 6 & K & 0 \\ 1 & 8 & 0 \\ 0 & 6 & K \end{vmatrix} = K(48 - K) > 0$$

Therefore, the system is always stable for $K < 48$.

Method-II,

Characteristics equation is $B(s) = s^3 + 6s^2 + 8s + K = 0$

andRouth's array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48-K}{6} & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array if $K < 48$. Therefore, the system is always stable for $0 < K < 48$.

6. Find stability of the following system given by $B(s) = s^3 + 5s^2 + 10s + 3$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given Characteristics equation is $B(s) = s^3 + 5s^2 + 10s + 3 = 0$, andRouth's array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 5 & 3 \\ s^1 & 9.4 & 0 \\ s^0 & 3 & \end{array}$$

There are no sign changes in first column elements of this array. Therefore, the system is always stable.

7. Find stability of the following system given by $B(s) = s^3 + 2s^2 + 3s + 10$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given characteristics equation is

$B(s) = s^3 + 2s^2 + 3s + 10 = 0$ and

Routh's array is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 2 & 10 \\ s^1 & -2 & 0 \\ s^0 & 10 & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

8. Examine stability of the following system given by $s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & 0 & 2.5 & \\ s^2 & \infty & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

Method-1

(i) Replace 0 by ε(very small number) and complete the array with ε.
(ii) Examine the sign change by taking $\varepsilon \to 0$

Now, Routh's array becomes

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & \varepsilon & 2.5 & 0 \\ s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\ s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\ s^0 & 1 & & \end{array}$$

Now putting $\varepsilon \to 0$, Routh's array becomes

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & \varepsilon & 2.5 & 0 \\ s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\ s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\ s^0 & 1 & & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Method-2

Replace s by $\frac{1}{Z}$. The system characteristic equation $s^5+2s^4+4s^3+8s^2+3s+1=0$ becomes

$$\frac{1}{Z^5}+\frac{2}{Z^4}+\frac{4}{Z^3}+\frac{8}{Z^2}+\frac{3}{Z}+1=0$$

$$\Rightarrow Z^5+3Z^4+8Z^3+4Z^2+2Z+1=0$$

Now, Routh's array becomes

$$\begin{array}{c|ccc} s^5 & 1 & 8 & 2 \\ s^4 & 3 & 4 & 1 \\ s^3 & 6.67 & 1.67 & 0 \\ s^2 & 3.25 & 1 & 0 \\ s^1 & -0.385 & 0 & 0 \\ s^0 & 1 & 0 & 0 \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

9. Examine stability of the following system given by $s^5+2s^4+2s^3+4s^2+4s+8$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 4 \\ s^4 & 2 & 4 & 8 \\ s^3 & 0 & 0 & 0 \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

The auxillary equation is

$$A(s) = 2s^4 + 4s^2 + 8$$

$$\Rightarrow \frac{dA(s)}{ds} = 8s^3 + 8s$$

Now, the array is rewritten as follows.

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 4 \\ s^4 & 2 & 4 & 8 \\ s^3 & 8 & 8 & 0 \\ s^2 & 2 & 8 & 0 \\ s^1 & -24 & 0 & \\ s^0 & 8 & & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

10. Examine stability of the following system given by $s^4 + 5s^3 + 2s^2 + 3s + 1 = 0$ using Routh-Hurwitz stability criterion. Find the number of roots in the right half of the s-plane.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 5 & 3 & 0 \\ s^2 & 1.4 & 2 & \\ s^1 & -4.14 & 0 & \\ s^0 & 2 & & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable. There are two poles in the right half of the s-plane.

2.4.3. Advantages of Routh-Hurwitz stability

(i) Stability can be judged without solving the characteristic equation
(ii) Less calculation time
(iii) The number of roots in RHP can be found in case of unstable condition
(iv) Range of value of K for system stability can be calculated
(v) Intersection point with the jw-axis can be calculated
(vi) Frequency of oscillation at steady-state is calculated

2.4.4. Advantages of Routh-Hurwitz stability

(i) It is valid for only real coefficient of the characteristic equation
(ii) Unable to give exact locations of closed-loop poles
(iii) Does not suggest methods for stabilizing an unstable system
(iv) Applicable only to the linear system

Equation Chapter 1 Section 1

MODULE#3

Equation Chapter (Next) Section 1

CHAPTER#10

10. Root locus

10.1. Definition:

The locus of all the closed-loop poles for various values of the open-loop gain K is called **root locus**. The root-locus method is developed by W.R. Evans in 1954. It helps to visualize the various possibilities of transient response of stable systems.

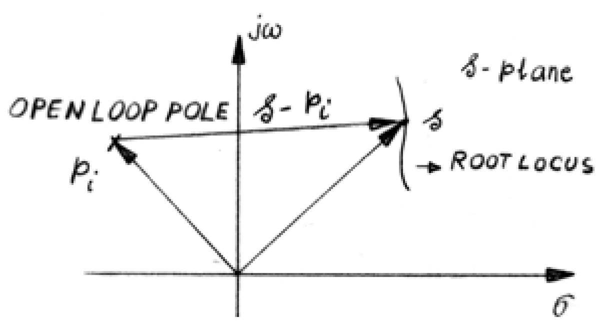
Closed-loop response function

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)} \tag{10.1}$$

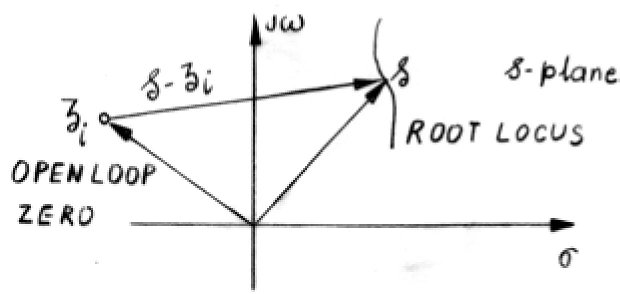
Characteristic equation

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0 \tag{10.2}$$

Vector from open-loop pole to the root-locus



Vector from open-loop zero to the root-locus



Behaviors of closed-loop poles

Closed-loop poles negative and real	Exponential decay	Stable
Closed-loop poles complex with negative real parts	Decaying and oscillatory	Stable
Closed-loop poles positive and real	Exponential increase	Unstable
Closed-loop poles complex with positive real parts	Exponential and oscillatory increase	Unstable

10.2. BASIS for CONSTRUCTION

10.2.1. Construction steps

1. Determine the number of open-loop poles and zeros
2. Mark open-loop poles and zeros on the s-plane
3. Determine parts of the root-locus on the real axis
4. Determine breakaway and break-in points
5. Draw asymptotes to the root-locus
6. Determine angles of departure
7. Determine angles of arrival
8. Determine points on the root-locus crossing imaginary axis
9. Obtain additional points and complete the root-locus

10.2.2. Starting points

Characteristics equation of a closed-loop system

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0 \quad (10.3)$$

For K=0,

$$\Rightarrow \frac{(s-p_1)(s-p_2)...(s-p_n)+K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0$$

$$\Rightarrow (s-p_1)(s-p_2)...(s-p_n)=0 \quad (10.4)$$

Open-loop poles are also closed-loop poles for K=0. A root-locus starts from every open-loop pole.

10.2.3. Ending points

Characteristics equation of a closed-loop system

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0 \quad (10.5)$$

For K=∞,

$$1<<\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

$$\Rightarrow (s-z_1)(s-z_2)...(s-z_m)=0 \quad (10.6)$$

Root-locus ends at an open-loop zero or at infinity.

10.2.4. Magnitude and angle criterion

$$1+G(s)H(s)=1+\left|G(s)H(s)\right|(\cos\psi + j\sin\psi)=0 \quad (10.7)$$

Angle criterion:

$$\psi=\sum_{i=1}^{n}\theta_i-\sum_{j=1}^{m}\varphi_{j=}180^0 \pm 360k \quad (10.8)$$

Where, $\theta_i =$ angle in case of ith pole and $\varphi_{j=}$ angle in case of jth zero

Magnitude criterion:

$$\left|G(s)H(s)\right| = 1 \tag{10.9}$$

10.2.5. Determining gain at a root-locus point

Using the magnitude of vectors drawn from open-loop poles and zeros to the root-locus point, we get

$$\frac{\prod_{i=1}^{n}(s-p_i)}{\prod_{j=1}^{m}(s-z_j)} = \frac{|(s-p_1)||(s-p_2)|\ldots|(s-p_n)|}{|(s-z_1)||s-z_2)|\ldots|(s-z_m)|} = K \tag{10.10}$$

Gain at a root-locuspoint is determinedusing synthetic division.

Example:
Determine K of the characteristic equation for the root s=-0.85.

Solution:

$$S^3 + 6s^2 + 8s + K = 0 \tag{10.11}$$

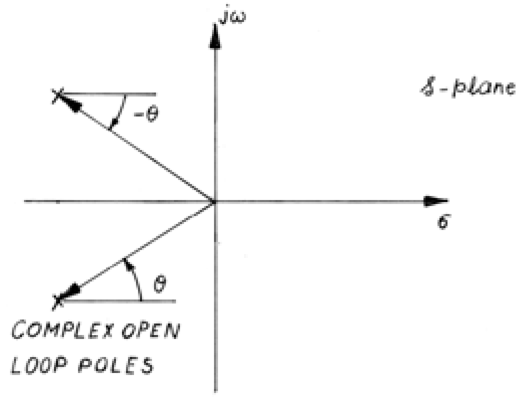
1	6	8	K
	-0.85	-4.378	-3.079
1	5.15	3.622	K-3.079=0

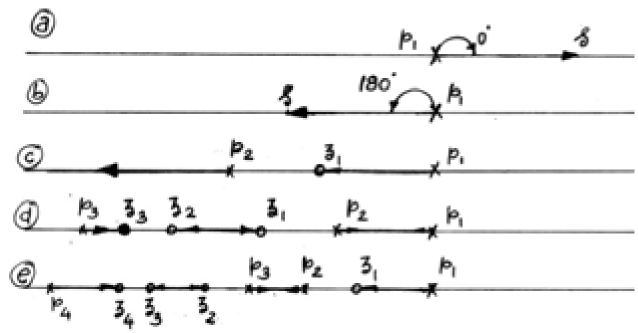
10.2.6. Determine parts of the root-locus on the real axis

1. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an open-loop zero on the real axis.
2. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an infinite value on the real axis.
3. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis and end at a complex open-loop zero or infinity.
4. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis. They may once again enter the real axis and end at open-loop zeros or at a large value on the real axis.
5. Start from a pair of complex open-loop poles, enter the real axis and end at an open-loop zero or an infinite value on the real axis. They could leave the real axis again and end at a complex open-loop zero or infinity.

10.2.7. Angle contributions from complex poles

Complex poles and zeros do not contribute to the angle criterion on the real axis





10.2.8. Determine breakaway and break-in points of the root-locus

$$1+G(s)H(s)=1+K\frac{A(s)}{B(s)}=0 \tag{10.12}$$

$$f(s)=B(s)+KA(s)=0 \tag{10.13}$$

$$K=-\frac{B(s)}{As)} \tag{10.14}$$

$$f(s)=(s-s_1)^r(s-s_2)...(s-s_{n-r+1})=0 \tag{10.15}$$

$$\frac{df(s)}{ds}=r(s-s_1)^{r-1}(s-s_2)...(s-s_{n-r+1})+(s-s_1)^r.(s-s_3)..(s-s_{n-r+1})+... \tag{10.16}$$

$$\left.\frac{df(s)}{ds}\right|_{s=s_1}=0 \tag{10.17}$$

$$f^{'}(s)=B^{'}(s)+KA^{'}(s)=0 \tag{10.18}$$

$$\Rightarrow K=-\frac{B^{'}(s)}{A^{'}(s)} \tag{10.19}$$

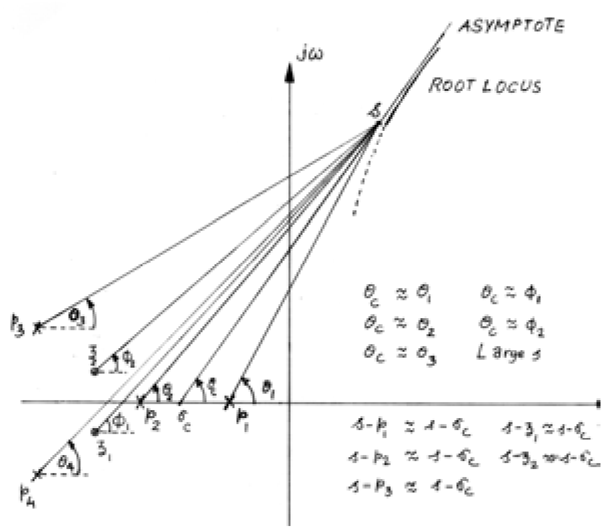
Therefore,

$$B^{'}(s)A(s)-B(s)A^{'}(s)=0 \tag{10.20}$$

At breakaway and break-in points of the root-locus,

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0 \quad (10.21)$$

10.2.9. Draw asymptotes to the root-locus



Angle of asymptotes

$\theta_c = \frac{180^0 + k360}{(n-m)}$ where, *k=0, 1, 2, 3..*

Location of asymptotes

$$-K = \frac{(s-p_1)(s-p_2)...(s-p_n)}{(s-z_1)(s-z_2)...(s-z_m)} \quad (10.22)$$

$$-K = \frac{s^n - (p_1 + p_2 + ...p_n)s^{n-1} + \ldots}{s^m - (z_1 + z_2 + ...z_m)s^{m-1} + ...} \quad (10.23)$$

$$-K = s^{n-m} - [(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)]s^{n-m-1} + ... \quad (10.24)$$

$$s - p_i \approx s - \sigma_c \quad (10.25)$$

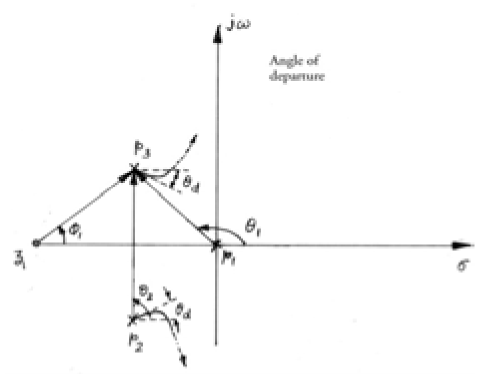
$$(s - z_i) \approx s - \sigma_c \quad (10.26)$$

$$-K = \frac{(s-\sigma_c)^n}{(s-\sigma_c)^m} = s^{n-m} - (n-m)\sigma_c s^{n-m-1} + ... \quad (10.27)$$

$$\sigma_c = \frac{(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)}{(n-m)} \quad (10.28)$$

Angle of departure

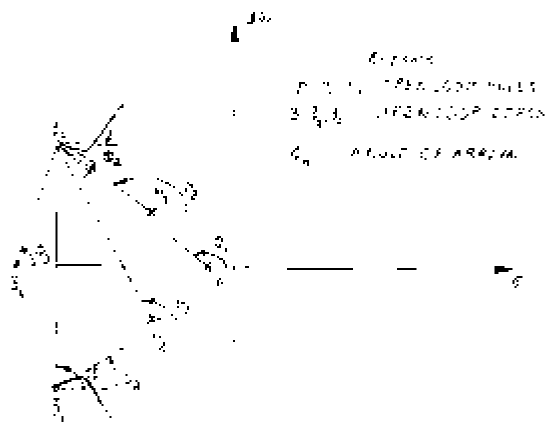
$$\theta_d = 180 - (\theta_1 + \theta_2) + \varphi_1 \quad (10.29)$$



$\theta_d = 180^o$ - ∑angles of vectors to the complex open-loop pole in question from other open - loop poles + ∑angles of vectors to the complex open-loop pole in question from all open-loop zeros

Angle of arrival

$$\theta_a = 180 - (\phi_1 + \phi_3) + (\theta_1 + \theta_2 + \theta_3) \tag{10.30}$$



$\theta_a = 180^o$ - ∑angles of vectors to the complex open-loop zero in question from other open- loop zeros + ∑angles of vectors to the complex open-loop zero in question from all open-loop poles

Determine points on the root-locus crossing imaginary axis

$$\mathrm{Re}\,al[1 + G(j\omega)H(j\omega)] = 0 \tag{10.31}$$

$$imaginary[1 + G(j\omega)H(j\omega)] = 0 \tag{10.32}$$

Example

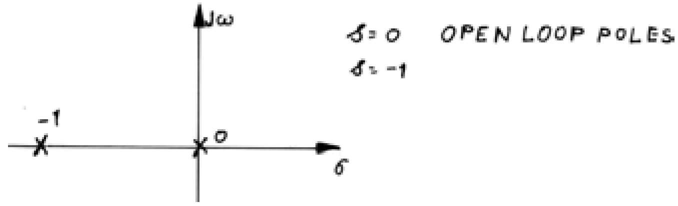
Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s(s+1)}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

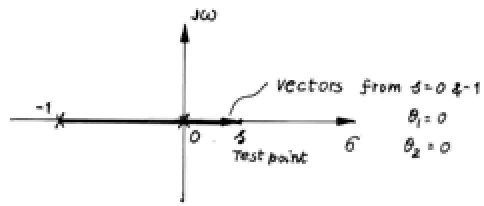
Number of open-loop poles n=2
Number of open-loop zeros m=0
Open-loop poles: s=0 and s=-1

Step 2: Mark open-loop poles and zeros on the s-plane

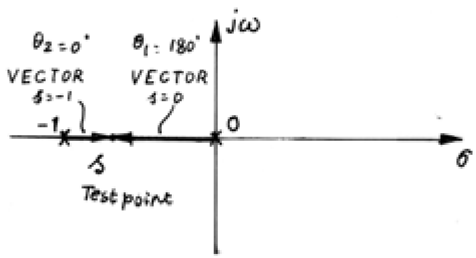


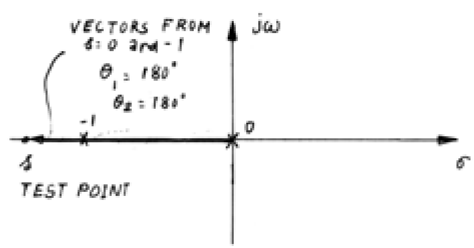
Step 3: Determine parts of the root-locus on the real axis

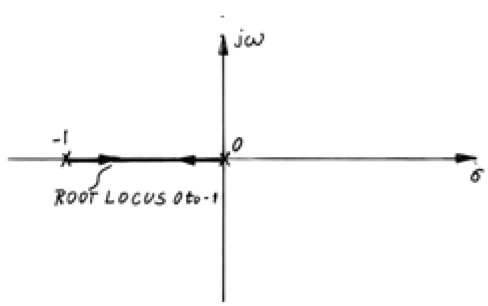
Test points on the positive real axis



Test points in between the open-lop poles







Step 4: Determine breakaway and break-in point

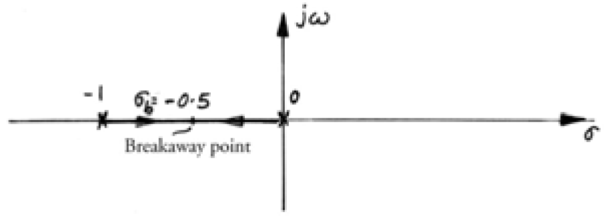
Characteristic equation, $K = -s(s+1)$

$$\frac{dK}{ds} = -2s + 1 = 0$$

breakaway point as σ_b=-0.5

Gain at the breakaway point

$K_b = |-0.5-0||-0.5-(-1)| = 0.25$



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

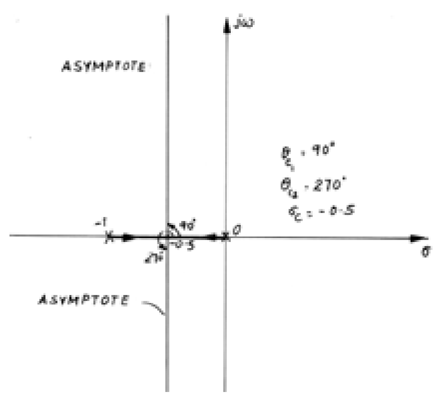
$$\theta_c = \theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{2}$$

$$\theta_c = 90^0 \;\; k = 0$$

$$\theta_c = 270^0 \;\; k = 1$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)}{(n-m)} = \frac{0-1}{2} = -0.5$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

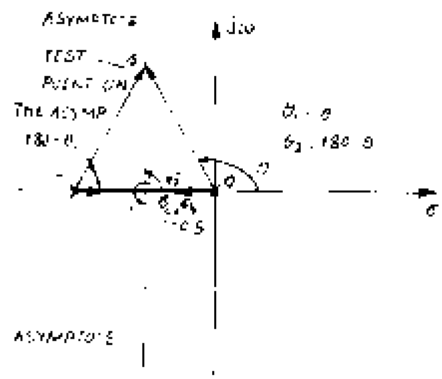
Step 8: Determine points on the root-locus crossing imaginary axis

$$1 + GH = 1 + \frac{K}{s(s+1)} = s^2 + s + K = 0$$

$$B(j\omega) = (j\omega)^2 + (j\omega) + K = (K - \omega^2) + j\omega$$

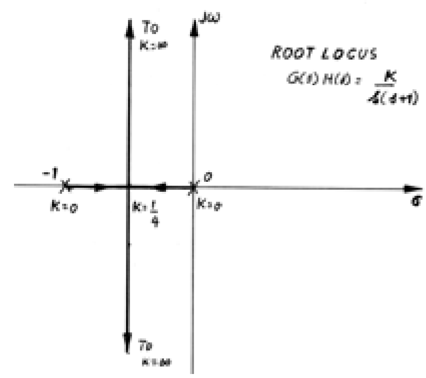
$$K - \omega^2 = 0 \Rightarrow j\omega = 0$$

The root-locus does not cross the imaginary axis for any value of K>0



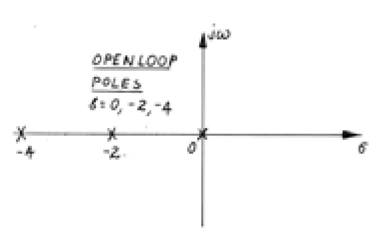
Here,

$$s = \frac{-1 \pm \sqrt{1-4K}}{2}$$

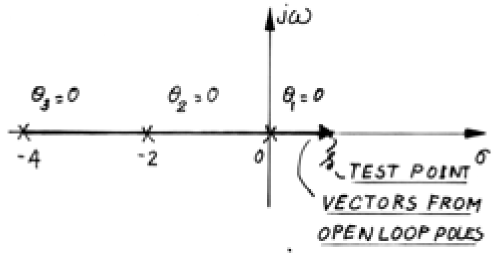


Problem-2: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$

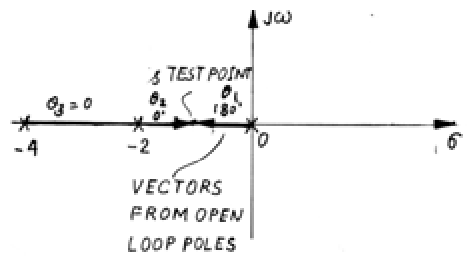
Solution:
Step 1: Determine the number of open-loop poles and zeros
Number of open-loop poles n=3
Number of open-loop zeros m=0
Open-loop poles: s=0, s=-2 and s=-4
Step 2: Mark open-loop poles and zeros on the s-plane

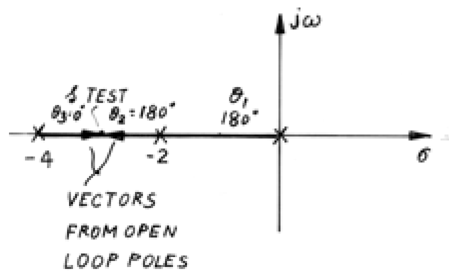


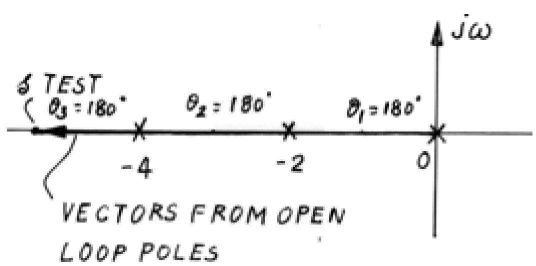
Step 3: Determine parts of the root-locus on the real axis
Test points on the positive real axis

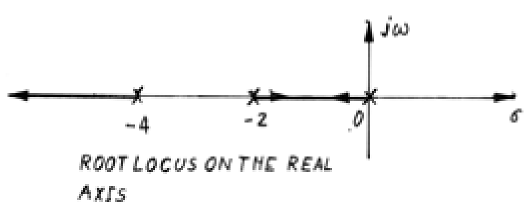


Test points in between the open-lop poles









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -s(s+2)(s+4)$

$$\frac{dK}{ds} = -(s+2)(s+4) - s(s+4) - s(s+2) = 0$$

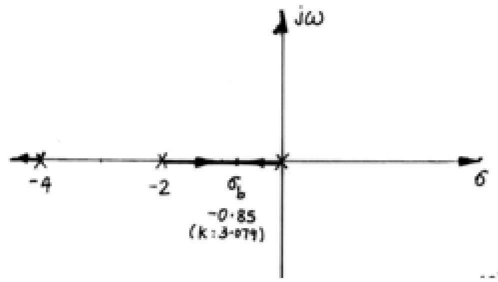
Breakaway point as σ_b=-0.85 and –3.15

σ_b = –3.15 is not on the root-locus and therefore not a breakaway or break-in point

Gain at the breakaway point

$K_b = |-0.85 - 0||-0.855 - (-2)||-0.85 - (-4)| = 3.079$

1	6	8	K
	-0.85	-4.378	-3.079
1	5.15	3.622	K-3.079=0



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{3}$$

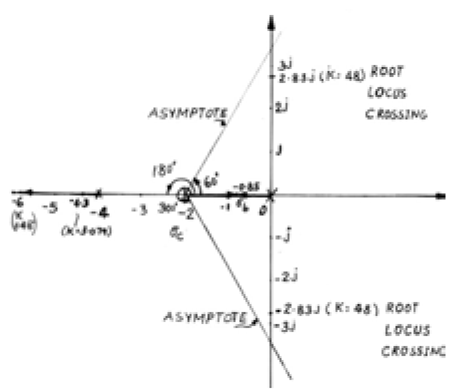
$$\theta_c = 60^0 \;\; k = 0$$

$$\theta_c = 180^0 \;\; k = 1$$

$$\theta_c = 300^0 \;\; k = 2$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \ldots p_n) - (z_1 + z_2 + \ldots z_m)}{(n-m)} = \frac{0-2-4}{3} = -2$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

Step 8: Determine points on the root-locus crossing imaginary axis

$$1 + GH = 1 + \frac{K}{s(s+2)(s+4)} = s^3 + 6s^2 + 8s + K = 0$$

$$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = (K - 6\omega^2) + j(8\omega - \omega^3) = 0$$

When imaginary-part is zero, then $\omega = \pm\sqrt{8} \Rightarrow s = \pm j\sqrt{8}$ and when real-part is zero, then $K = 6\omega^2 = 48$.
The root-locus does not cross the imaginary axis for any value of K>48.

1	6	8	48
	+j2.828	-8+j16.97	-48
1	6+j2.828	J16.97	0
1	6+j2.828	J16.97	
	-j2.828	-j16.97	
1	6	0	

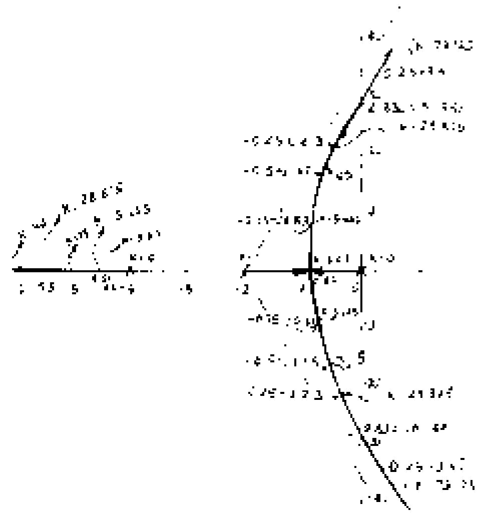
Therefore, closed-loop pole on the real axis for K=48 at $s = -6$

No.	**Closed-loop pole on the real axis**	**K**	**Second and third closed- loop poles**	**Remarks**
1	-4.309	3.07	-0.85,-0.85	Already computed
2	-4.50	5.625	-0.75±j0.829	
3	-5.00	15	-0.5±j1.6583	
4	-5.50	28.875	-0.25±j2.2776	
5	-6.00	48	±j2.8284	Already computed
6	-6.5	73.125	0.25±j3.448	

Determine the gain corresponding to s=-4.5
K=|-4.5-(-4)||-4.5-(-2)||-4.5-0|= 5.625
$s^3 + 6s^2 + 8s + K = 0$

1	6	8	K
	-4.5	-6.75	-5.625
1	1.5	1.25	K-5.625=0

$(s^2 + 1.5s + 1.25) = 0$
$s_{2,3} = -0.75 \pm j0.829$



Problem-3: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s^2(s+1)}$

Solution:

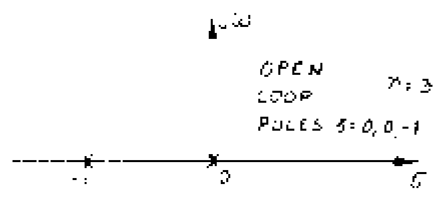
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=3

Number of open-loop zeros m=0

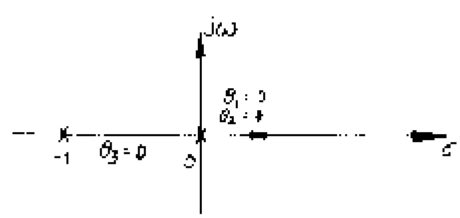
Open-loop poles: s=0, s=0 and s=-1

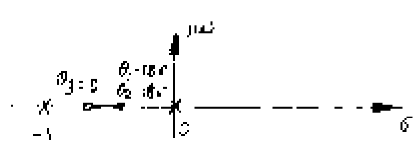
Step 2: Mark open-loop poles and zeros on the s-plane

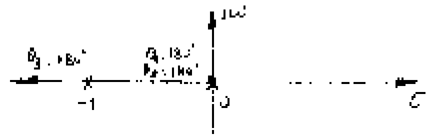


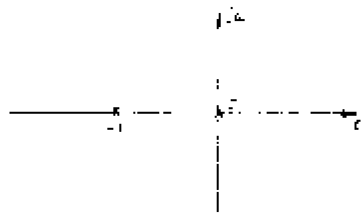
Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -s^2(s+1)$

$$\frac{dK}{ds} = 0$$

$$\Rightarrow -2s(s+1) - s = 0$$

$$\Rightarrow s(-2s-3) = 0$$

Breakaway point as σ_b= -2/3and 0

σ_b = -2/3is not on the root-locus and therefore not a breakaway or break-in point.

Therefore σ_b= 0 and the two loci start from the origin and breakaway at the origin itself.

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{3}$$

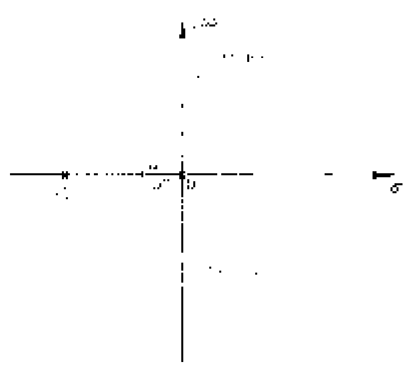
$$\theta_c = 60^0 \;\; k = 0$$

$$\theta_c = 180^0 \;\; k = 1$$

$$\theta_c = 300^0 \;\; k = 2$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \ldots p_n) - (z_1 + z_2 + \ldots z_m)}{(n-m)} = \frac{0-1}{3} = -\frac{1}{3}$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s) = s^3 + s^2 + K$

$B(j\omega) = (j\omega)^3 + (j\omega)^2 + K = (K - \omega^2) - j\omega^3$

When imaginary-part is zero, then $\omega = 0 \Rightarrow s = 0$ and when real-part is zero, then $K = \omega^2 = 0$.

The root-locus does not cross the imaginary axis for any value of K>0.

Additional closed-loop poles

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles
1	-1.5	1.125	0.25±j0.82
2	-2.0	4	0.50±j1.32
3	-2.5	9.375	0.75±j1.78
4	-3.0	18	1.00±j2.23

Determine the gain corresponding to s=-1.5

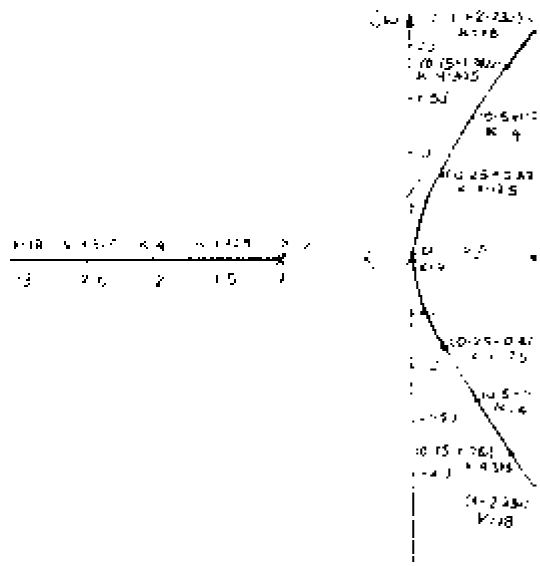
K=|-1.5-(-1)||-1.5-(0)||-1.5-0|= 1.125

$s^3 + s^2 + 1.125 = 0$

1	1	0	1.125
	-1.5	0.75	-1.125
1	-0.5	0.75	0

$(s^2 + 1.5s + 1.25) = 0$

$s_{2,3} = -0.25 \pm j0.82$



Problem-4: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s^4 + 5s^3 + 8s^2 + 6s}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

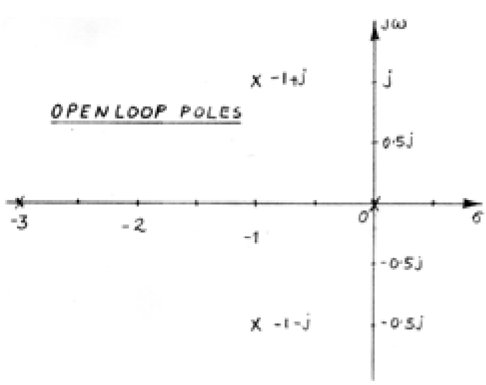
$s^4 + 5s^3 + 8s^2 + 6s = s(s^2 + 2s + 2)(s+3) = (s+1-j)(s+1+j)(s+3)s$

Number of open-loop poles n=4

Number of open-loop zeros m=0

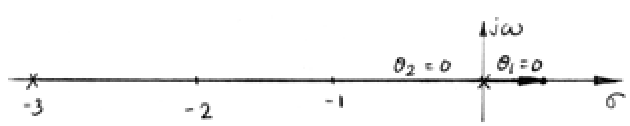
Open-loop poles: s=0 and s=-3, s=-1+j and s=-1-j

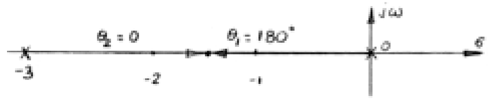
Step 2: Mark open-loop poles and zeros on the s-plane

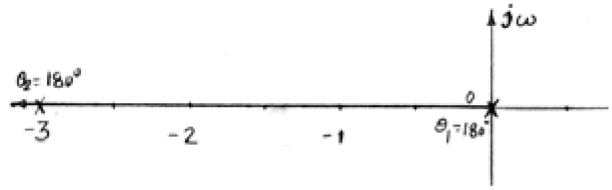


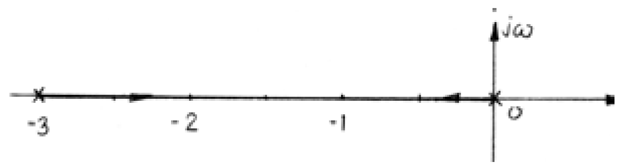
Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -(s^4 + 5s^3 + 8s^2 + 6s)$

$$\frac{dK}{ds} = 0$$

$$\Rightarrow 4s^3 + 15s^2 + 16s + 6 = 0$$

$$\Rightarrow s^3 + 3.75s^2 + 4s + 1.5 = 0$$

$$f'(s) = 3s^2 + 7.5s + 4$$

This equation is solved using Newton-Raphson's method

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)}$$

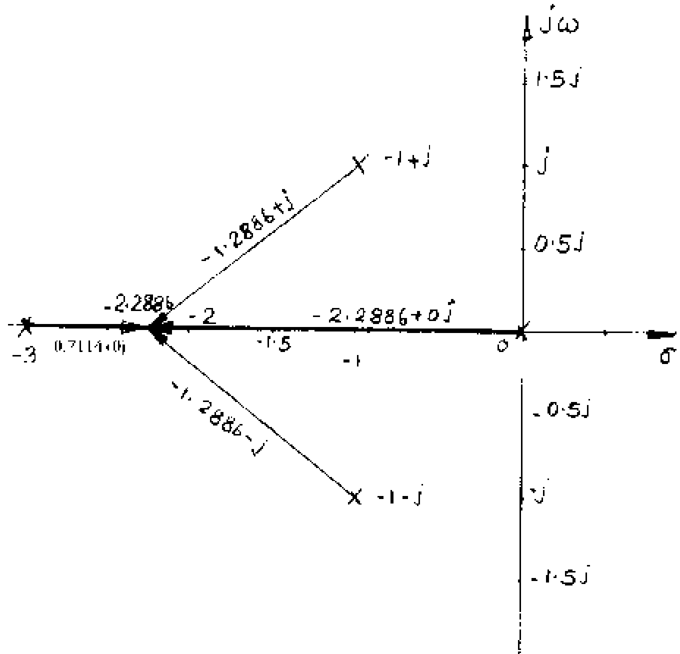
No.	s_n	$f(s_n)$	$f'(s_n)$	s_{n+1}
1	-3.75	-13.5	18.0625	-3.0026
2	-3.0026	-3.7721	8.5273	-2.5602
3	-2.5602	-0.9421	4.4624	-2.3491
4	-2.3491	-0.1658	2.9364	-2.2926
5	-2.2926	-0.0103	2.5737	-2.2886
6	-2.2886	-5.03×10^{-5}		

Breakaway point as σ_b= -2.3

Gain at the breakaway point, $K = |-2.3-(-3)||-2.3-0||-2.3-(-1+j)||-2.3-(-1-j)| = 4.33$

1	5	8	6	K
	-2.2886	-6.2053	-4.1073	-4.3316

1	2.7114	1.7947	1.8926	0



Other closed-loop poles for K=4.3

1	2.7114	1.7947	1.893
	-2.2886	-0.9676	-1.893
1	0.4228	0.8270	0

$s_{3,4}$ =-0.2114±j0.8814

Step 5: Draw asymptotes of the root-locus
Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{4}$$

$$\theta_c = 45^0 \;\; k = 0$$

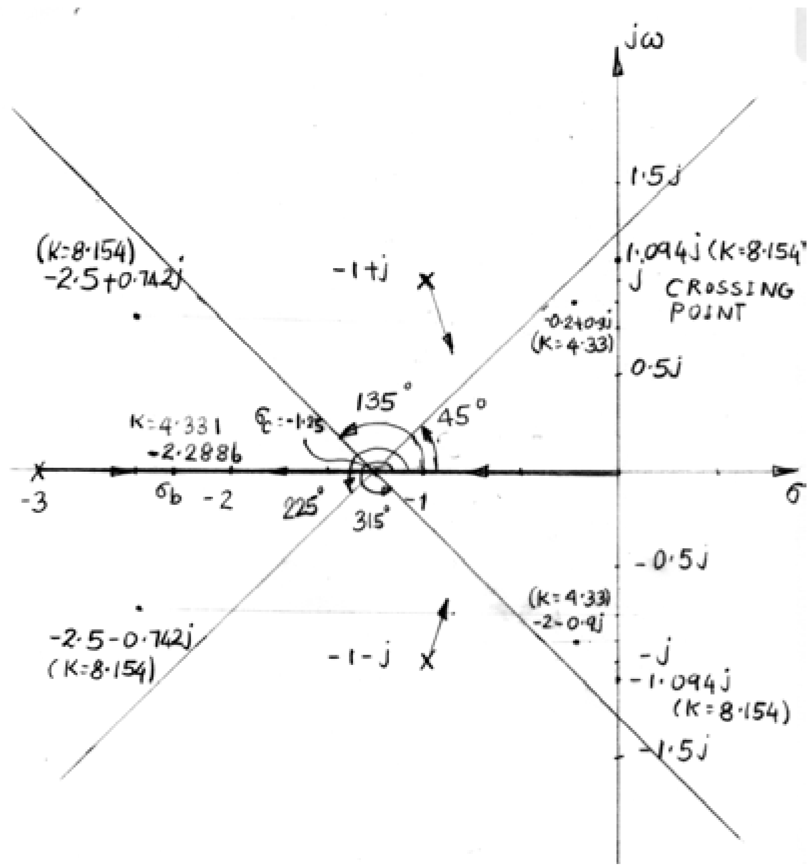
$$\theta_c = 135^0 \;\; k = 1$$

$$\theta_c = 225^0 \;\; k = 2$$

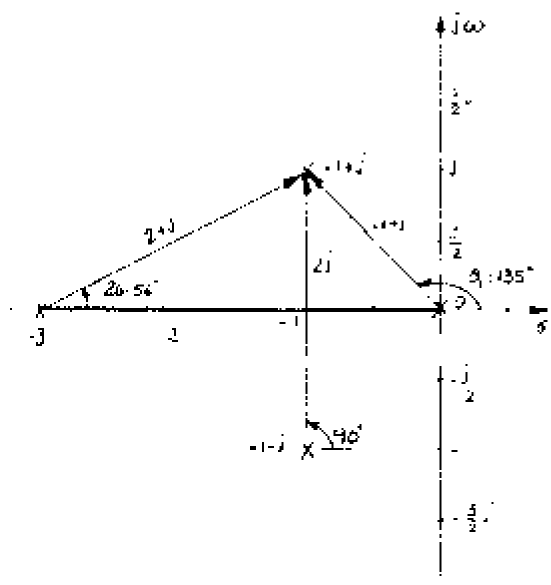
$$\theta_c = 315^0 \;\; k = 3$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \dots p_n) - (z_1 + z_2 + \dots z_m)}{(n-m)} = \frac{0 - 3 - 1 + j - 1 - j}{4} = -1.25$$



Steps 6: Determine angles of departure



$\theta_d = 180^0 - (135^0 + 26.56^0 + 90^0) = -71.56^0 = 288.44^0$

Step 7: As there are no complex open-loop zeros, angle of arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s) = s^4 + 5s^3 + 8s^2 + 6s + K$

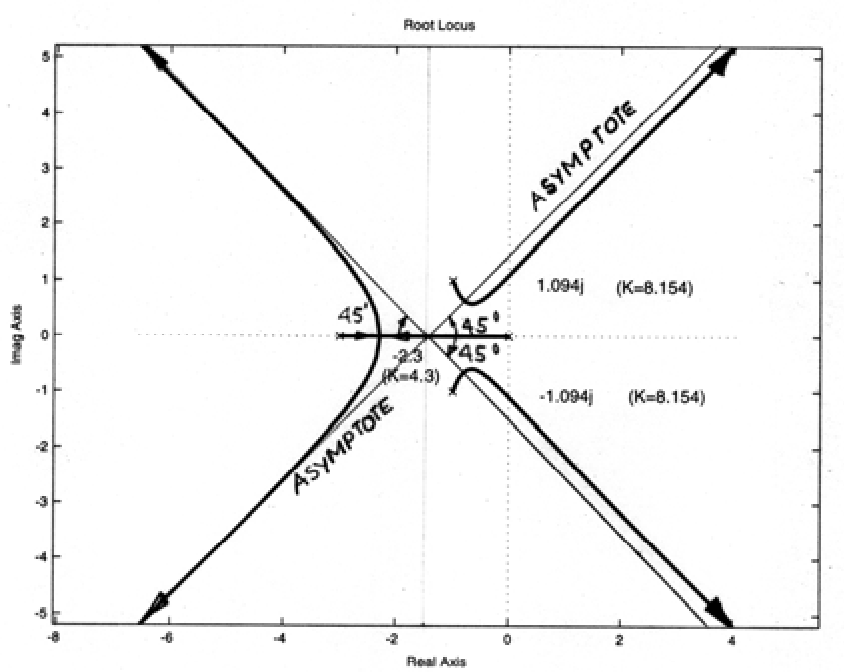
$B(j\omega) = (j\omega)^4 + 5(j\omega)^3 + 8(j\omega)^2 + 6j\omega + K = (\omega^4 - 8\omega^2 + K) + j(6\omega - 5\omega^3)$

When imaginary-part is zero, then $\omega = \pm\sqrt{\frac{6}{5}} \Rightarrow s = \pm j\sqrt{\frac{6}{5}}$ and when real-part is zero,

then $K = 8\times\left(\frac{6}{5}\right) - \left(\frac{6}{5}\right)^2 = 8.16$.

There are two closed-loop poles on the imaginary axis for any value of K>0.
Additional closed-loop poles

No.	S_1	S_2	$S_{3,4}$	K
1	-0.25	-2.9217	-0.9142±0.7969j	1.0742
2	-0.50	-2.8804	-0.8098±0.655j	1.5625
3	-0.75	-2.8593	-0.6953±0.5938j	1.7930
4	-1.0	-2.8393	-0.5804±0.6063j	2.0000
5	-1.25	-2.8055	-0.4722±0.6631j	2.3242
6	-1.75	-2.6562	-0.3763±0.7354j	2.8125
7	-2.0	-2.5214	-0.2393±0.8579j	4.0



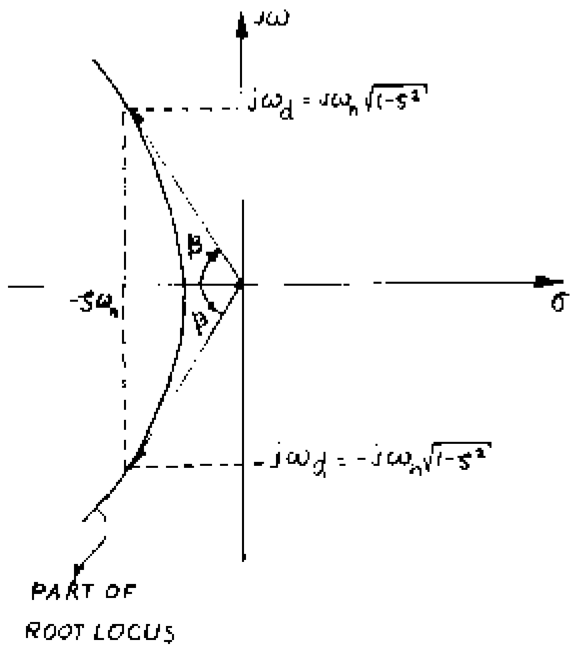
Additional Information from Root-Locus Plot

1. Gain Margin

$$GM = 20\log\frac{K_2}{K_1} \tag{10.33}$$

K_1 is the gain of a feedback system at some point on the root-locus
K_2 is the gain at which the system becomes unstable

2. Transient Characteristics



Where, $\beta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

3. Percentage overshoot

$$M_p = e^{-\pi/\tan\beta} \tag{10.34}$$

4. Settling time

$$t_s = \frac{4}{\zeta\omega_n} \tag{10.35}$$

5. Steady-state error is also related to K.

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \frac{K\left(s^2+10s+100\right)}{s^4+20s^3+100s^2+500s+1500}, H\left(s\right)=1$

(a) Determine the value of gain at which the system will be stable and as well have a maximum overshoot of 5%.

(b) What is the gain margin at this point?

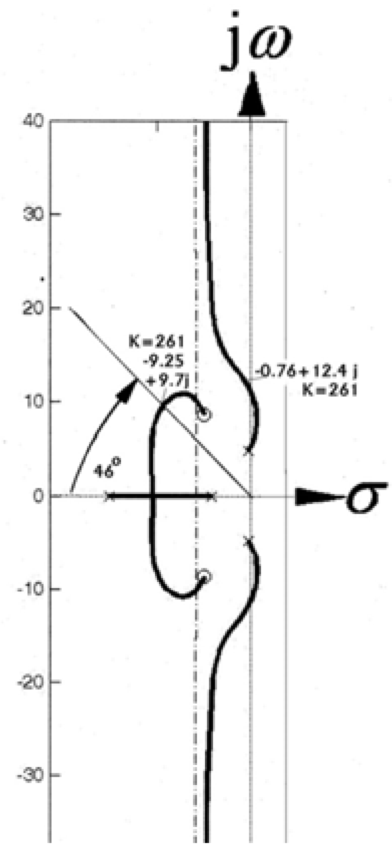
(c) What is the steady-state error for a unit step excitation at the above point?

Solution:

(a) $\tan\beta = \frac{-\pi}{\ln M_p} = 1.0487$

$\Rightarrow \beta = 46^0$

$$\zeta = \frac{1}{\sqrt{1+\tan^2\beta}} = 0.690 \qquad (10.36)$$



(b) $GM = 20\log\frac{192.2}{261} = -2.65dB$

(c) Position error

$$K_s = \lim_{s\to 0}\frac{K(s^2+10s+100)}{s^4+20s^3+100s^2+500s+1500} = \frac{100K}{1500}$$

Steady-state error,

$$S_e(\infty) = \frac{1}{1+K_s} = \frac{1}{1+100K/1500} = \frac{1500}{1500+100K}$$

$$S_e(\infty) = \frac{1500}{1500+100\times 261} = 5.4\%$$

Root locus

The locus of all the closed-loop poles for various values of the open-loop gain K is called **root locus**. The root-locus method is developed by W.R. Evans in 1954. It helps to visualize the various possibilities of transient response of stable systems.

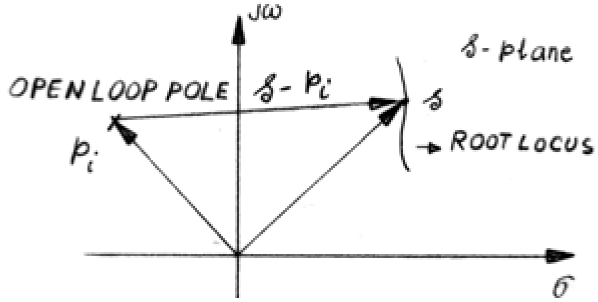
Closed-loop response function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{10.37}$$

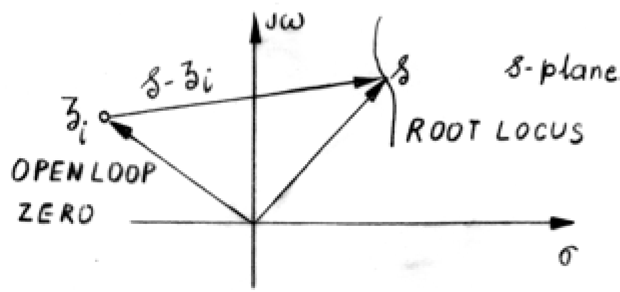
Characteristic equation

$$1+G(s)H(s) = 1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} = 0 \tag{10.38}$$

Vector from open-loop pole to the root-locus



Vector from open-loop zero to the root-locus



Behaviors of closed-loop poles

Closed-loop poles negative and real	Exponential decay	Stable
Closed-loop poles complex with negative real parts	Decaying and oscillatory	Stable
Closed-loop poles positive and real	Exponential increase	Unstable
Closed-loop poles complex with positive real parts	Exponential and oscillatory increase	Unstable

BASIS for CONSTRUCTION

Construction steps

10. Determine the number of open-loop poles and zeros

11. Mark open-loop poles and zeros on the s-plane
12. Determine parts of the root-locus on the real axis
13. Determine breakaway and break-in points
14. Draw asymptotes to the root-locus
15. Determine angles of departure
16. Determine angles of arrival
17. Determine points on the root-locus crossing imaginary axis
18. Obtain additional points and complete the root-locus

Starting points
Characteristics equation of a closed-loop system

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0 \qquad (10.39)$$

For K=0,

$$\Rightarrow \frac{(s-p_1)(s-p_2)...(s-p_n)+K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0$$

$$\Rightarrow (s-p_1)(s-p_2)...(s-p_n)=0 \qquad (10.40)$$

Open-loop poles are also closed-loop poles for K=0. A root-locus starts from every open-loop pole.

Ending points
Characteristics equation of a closed-loop system

$$1+G(s)H(s)=1+\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}=0 \qquad (10.41)$$

For K=∞,

$$1<<\frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

$$\Rightarrow (s-z_1)(s-z_2)...(s-z_m)=0 \qquad (10.42)$$

Root-locus ends at an open-loop zero or at infinity.

Magnitude and angle criterion

$$1+G(s)H(s)=1+\left|G(s)H(s)\right|(\cos\psi + j\sin\psi)=0 \qquad (10.43)$$

Angle criterion:

$$\psi=\sum_{i=1}^{n}\theta_i-\sum_{j=1}^{m}\varphi_{j=}180^0 \pm 360k \qquad (10.44)$$

Where, $\theta_i =$ angle in case of i^th^ pole and $\varphi_{j=}$ angle in case of j^th^ zero
Magnitude criterion:

$$\left|G(s)H(s)\right|=1 \qquad (10.45)$$

Determining gain at a root-locus point

Using the magnitude of vectors drawn from open-loop poles and zeros to the root-locus point, we get

$$\frac{\prod_{i=1}^{n}(s-p_i)}{\prod_{j=1}^{m}(s-z_j)} = \frac{|(s-p_1)||(s-p_2)|\ldots|(s-p_n)|}{|(s-z_1)||s-z_2)|\ldots|(s-z_m)|} = K \qquad (10.46)$$

Gain at a root-locus point is determined using synthetic division.

Example:
Determine K of the characteristic equation for the root s=-0.85.

Solution:

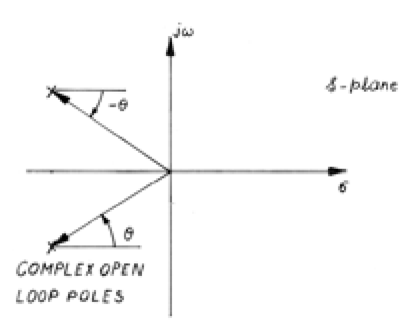
$$S^3 + 6s^2 + 8s + K = 0 \qquad (10.47)$$

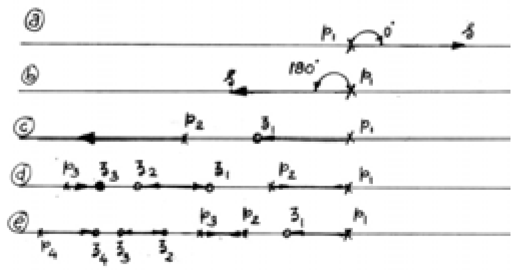
1	6	8	K
	-0.85	-4.378	-3.079
1	5.15	3.622	K-3.079=0

Determine parts of the root-locus on the real axis

6. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an open-loop zero on the real axis.
7. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an infinite value on the real axis.
8. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis and end at a complex open-loop zero or infinity.
9. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis. They may once again enter the real axis and end at open-loop zeros or at a large value on the real axis.
10. Start from a pair of complex open-loop poles, enter the real axis and end at an open-loop zero or an infinite value on the real axis. They could leave the real axis again and end at a complex open-loop zero or infinity.

Angle contributions from complex poles
Complex poles and zeros do not contribute to the angle criterion on the real axis





Determine breakaway and break-in points of the root-locus

$$1+G(s)H(s)=1+K\frac{A(s)}{B(s)}=0 \quad (10.48)$$

$$f(s)=B(s)+KA(s)=0 \quad (10.49)$$

$$K=-\frac{B(s)}{As)} \quad (10.50)$$

$$f(s)=(s-s_1)^r(s-s_2)...(s-s_{n-r+1})=0 \quad (10.51)$$

$$\frac{df(s)}{ds}=r(s-s_1)^{r-1}(s-s_2)...(s-s_{n-r+1})+(s-s_1)^r.(s-s_3)..(s-s_{n-r+1})+... \quad (10.52)$$

$$\left.\frac{df(s)}{ds}\right|_{s=s_1}=0 \quad (10.53)$$

$$f'(s)=B'(s)+KA'(s)=0 \quad (10.54)$$

$$\Rightarrow K=-\frac{B'(s)}{A'(s)} \quad (10.55)$$

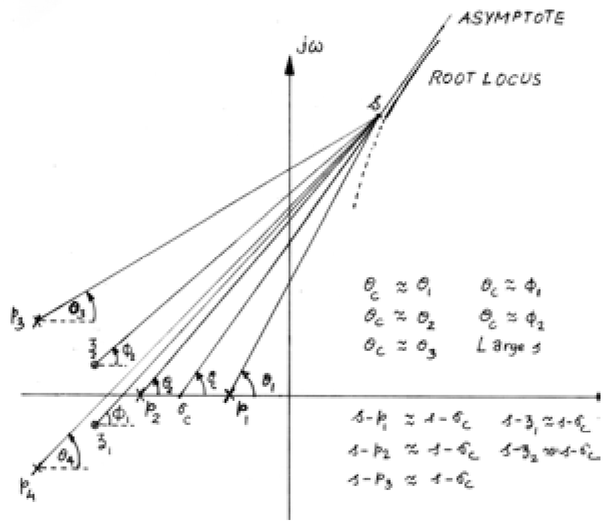
Therefore,

$$B'(s)A(s)-B(s)A'(s)=0 \quad (10.56)$$

At breakaway and break-in points of the root-locus,

$$\frac{dK}{ds}=-\frac{B'(s)A(s)-B(s)A'(s)}{A^2(s)}=0 \quad (10.57)$$

Draw asymptotes to the root-locus



Angle of asymptotes

$\theta_c = \dfrac{180^0 + k360}{(n-m)}$ where, $k=0, 1, 2, 3..$

Location of asymptotes

$$-K = \frac{(s-p_1)(s-p_2)...(s-p_n)}{(s-z_1)(s-z_2)...(s-z_m)} \tag{10.58}$$

$$-K = \frac{s^n - (p_1 + p_2 + ...p_n)s^{n-1} + \ldots}{s^m - (z_1 + z_2 + ...z_m)s^{m-1} + ...} \tag{10.59}$$

$$-K = s^{n-m} - [(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)]s^{n-m-1} + ... \tag{10.60}$$

$$s - p_i \approx s - \sigma_c \tag{10.61}$$

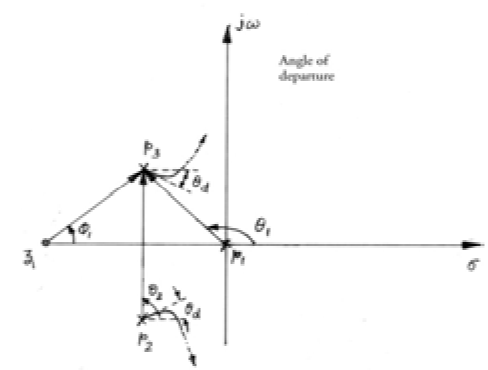
$$(s - z_i) \approx s - \sigma_c \tag{10.62}$$

$$-K = \frac{(s-\sigma_c)^n}{(s-\sigma_c)^m} = s^{n-m} - (n-m)\sigma_c s^{n-m-1} + ... \tag{10.63}$$

$$\sigma_c = \frac{(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)}{(n-m)} \tag{10.64}$$

Angle of departure

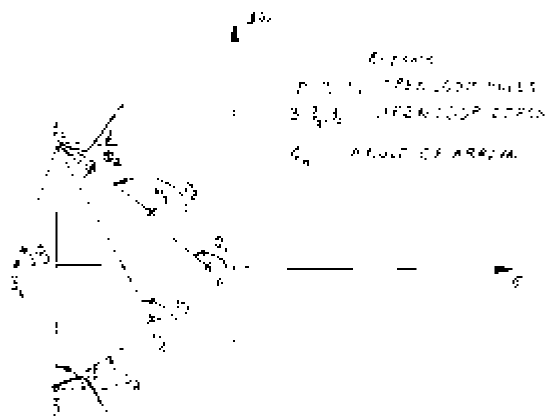
$$\theta_d = 180 - (\theta_1 + \theta_2) + \varphi_1 \tag{10.65}$$



$\theta_d = 180^o$ - ∑angles of vectors to the complex open-loop pole in question from other open - loop poles + ∑angles of vectors to the complex open-loop pole in question from all open-loop zeros

Angle of arrival

$$\theta_a = 180 - (\phi_1 + \phi_3) + (\theta_1 + \theta_2 + \theta_3) \tag{10.66}$$



$\theta_a = 180^o$ - ∑angles of vectors to the complex open-loop zero in question from other open- loop zeros + ∑angles of vectors to the complex open-loop zero in question from all open-loop poles

Determine points on the root-locus crossing imaginary axis

$$\mathrm{Re}al[1 + G(j\omega)H(j\omega)] = 0 \tag{10.67}$$

$$imaginary[1 + G(j\omega)H(j\omega)] = 0 \tag{10.68}$$

Example

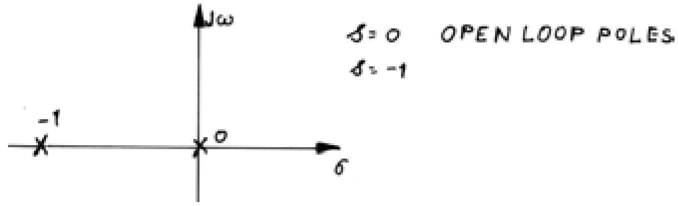
Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s(s+1)}$

Solution:

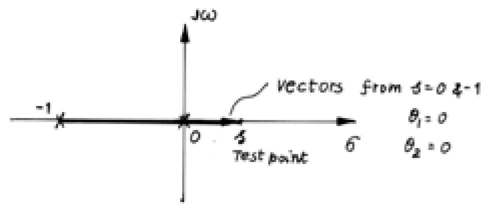
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=2
Number of open-loop zeros m=0
Open-loop poles: s=0 and s=-1

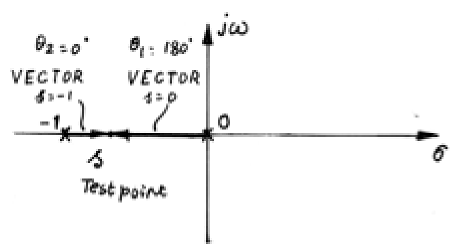
Step 2: Mark open-loop poles and zeros on the s-plane

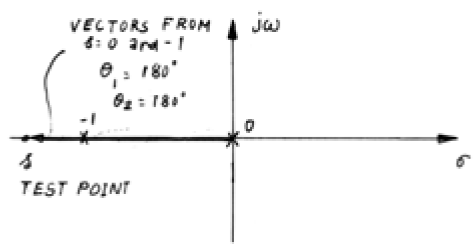


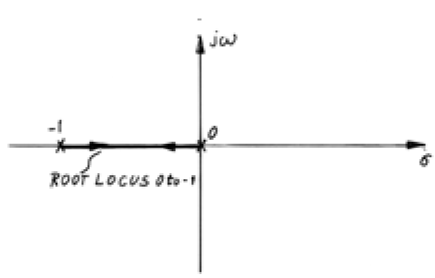
Step 3: Determine parts of the root-locus on the real axis
Test points on the positive real axis



Test points in between the open-lop poles





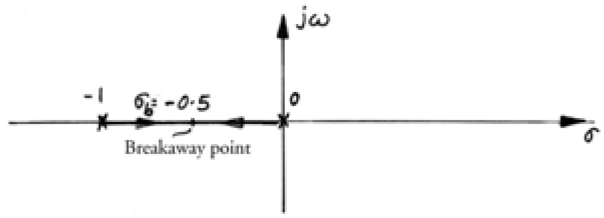


Step 4: Determine breakaway and break-in point
Characteristic equation, $K = -s(s+1)$

$$\frac{dK}{ds} = -2s + 1 = 0$$

breakaway point as σ_b=-0.5
Gain at the breakaway point

$K_b = |-0.5-0||-0.5-(-1)| = 0.25$



Step 5: Draw asymptotes of the root-locus
Angle of asymptotes:

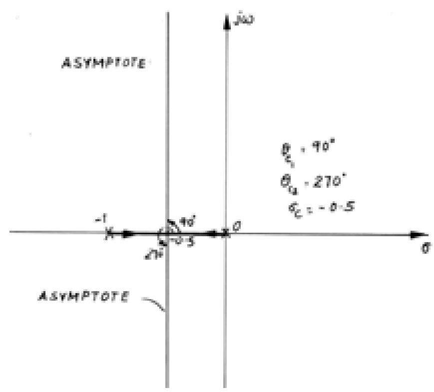
$$\theta_c = \theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{2}$$

$$\theta_c = 90^0 \ \ k=0$$

$$\theta_c = 270^0 \ \ k=1$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1+p_2+...p_n)-(z_1+z_2+...z_m)}{(n-m)} = \frac{0-1}{2} = -0.5$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

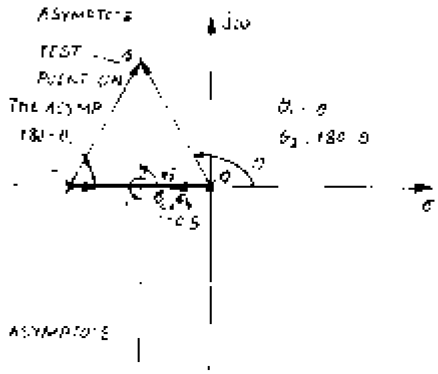
Step 8: Determine points on the root-locus crossing imaginary axis

$$1+GH = 1+\frac{K}{s(s+1)} = s^2+s+K = 0$$

$$B(j\omega) = (j\omega)^2 + (j\omega) + K = (K-\omega^2) + j\omega$$

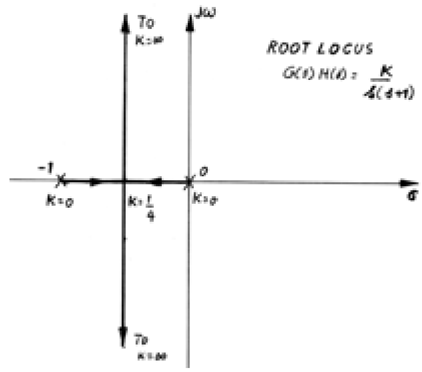
$$K-\omega^2 = 0 \Rightarrow j\omega = 0$$

The root-locus does not cross the imaginary axis for any value of K>0



Here,

$$s=\frac{-1\pm\sqrt{1-4K}}{2}$$



Problem-2: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s+2)(s+4)}$

Solution:

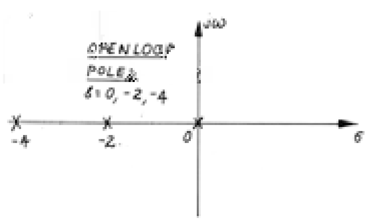
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=3

Number of open-loop zeros m=0

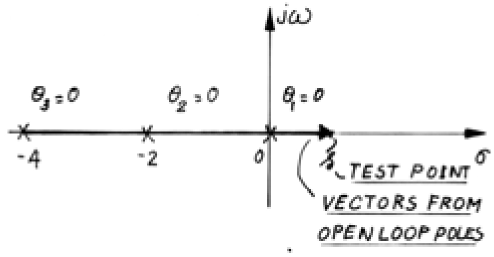
Open-loop poles: s=0, s=-2 and s=-4

Step 2: Mark open-loop poles and zeros on the s-plane

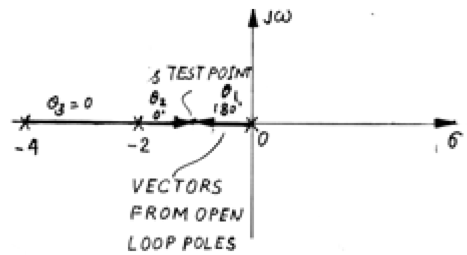


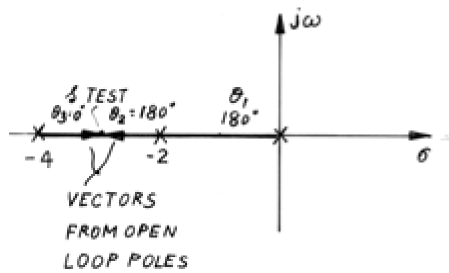
Step 3: Determine parts of the root-locus on the real axis

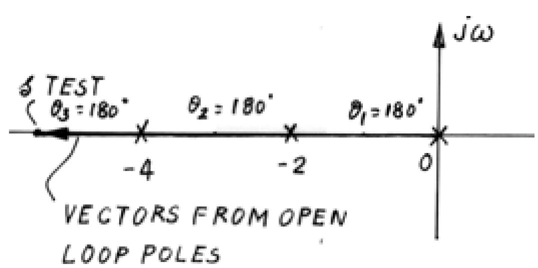
Test points on the positive real axis

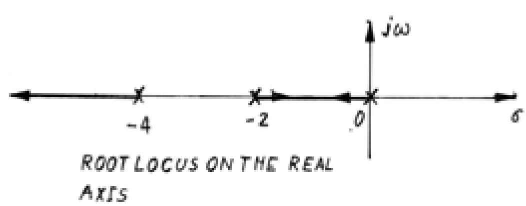


Test points in between the open-lop poles









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -s(s+2)(s+4)$

$$\frac{dK}{ds} = -(s+2)(s+4) - s(s+4) - s(s+2) = 0$$

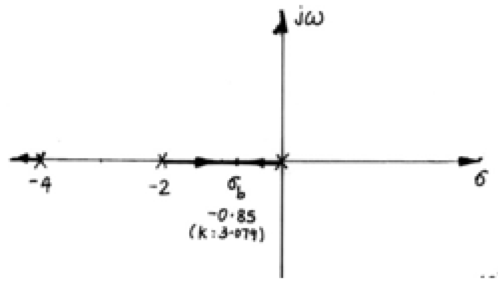
Breakaway point as σ_b=-0.85 and –3.15

σ_b = –3.15 is not on the root-locus and therefore not a breakaway or break-in point

Gain at the breakaway point

$$K_b = |-0.85-0||-0.855-(-2)||-0.85-(-4)| = 3.079$$

1	6	8	K
	-0.85	-4.378	-3.079
1	5.15	3.622	K-3.079=0



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{3}$$

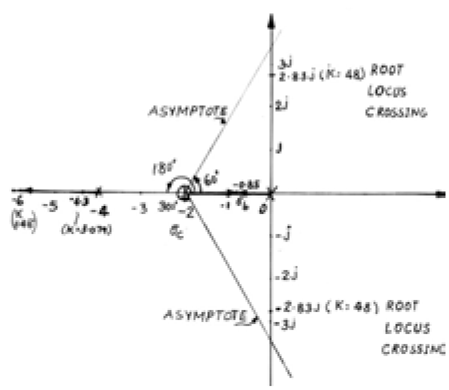
$$\theta_c = 60^0 \;\; k = 0$$

$$\theta_c = 180^0 \;\; k = 1$$

$$\theta_c = 300^0 \;\; k = 2$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \ldots p_n) - (z_1 + z_2 + \ldots z_m)}{(n-m)} = \frac{0-2-4}{3} = -2$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

Step 8: Determine points on the root-locus crossing imaginary axis

$$1 + GH = 1 + \frac{K}{s(s+2)(s+4)} = s^3 + 6s^2 + 8s + K = 0$$

$$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = (K - 6\omega^2) + j(8\omega - \omega^3) = 0$$

When imaginary-part is zero, then $\omega = \pm\sqrt{8} \Rightarrow s = \pm j\sqrt{8}$ and when real-part is zero, then $K = 6\omega^2 = 48$.
The root-locus does not cross the imaginary axis for any value of K>48.

1	6	8	48
	+j2.828	-8+j16.97	-48
1	6+j2.828	J16.97	0
1	6+j2.828	J16.97	
	-j2.828	-j16.97	
1	6	0	

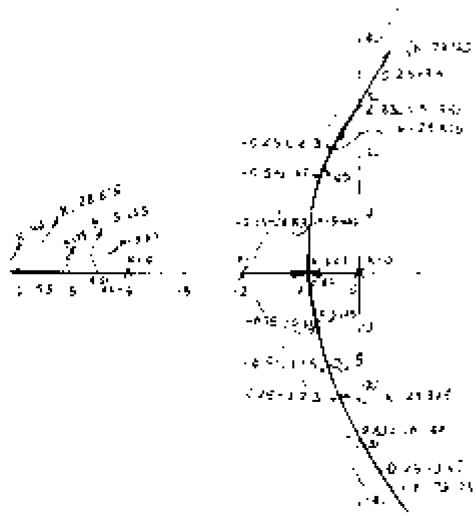
Therefore, closed-loop pole on the real axis for K=48 at $s = -6$

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles	Remarks
1	-4.309	3.07	-0.85,-0.85	Already computed
2	-4.50	5.625	-0.75±j0.829	
3	-5.00	15	-0.5±j1.6583	
4	-5.50	28.875	-0.25±j2.2776	
5	-6.00	48	±j2.8284	Already computed
6	-6.5	73.125	0.25±j3.448	

Determine the gain corresponding to s=-4.5
K=|-4.5-(-4)||-4.5-(-2)||-4.5-0|= 5.625
$s^3 + 6s^2 + 8s + K = 0$

1	6	8	K
	-4.5	-6.75	-5.625
1	1.5	1.25	K-5.625=0

$(s^2 + 1.5s + 1.25) = 0$
$s_{2,3} = -0.75 \pm j0.829$



Problem-3: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s^2(s+1)}$

Solution:

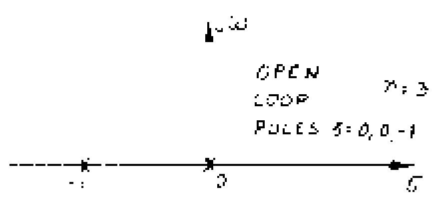
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=3

Number of open-loop zeros m=0

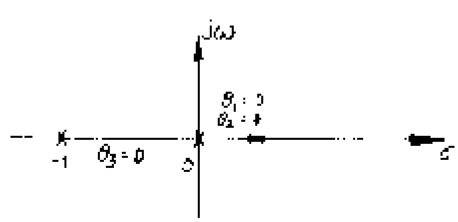
Open-loop poles: s=0, s=0 and s=-1

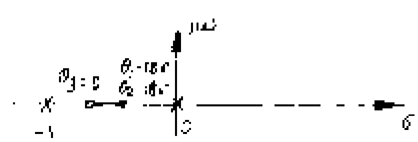
Step 2: Mark open-loop poles and zeros on the s-plane

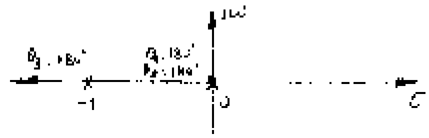


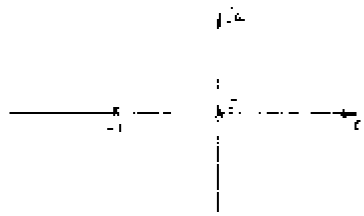
Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -s^2(s+1)$

$$\frac{dK}{ds} = 0$$

$$\Rightarrow -2s(s+1) - s = 0$$

$$\Rightarrow s(-2s-3) = 0$$

Breakaway point as σ_b= -2/3and 0

σ_b = -2/3is not on the root-locus and therefore not a breakaway or break-in point.

Therefore σ_b = 0 and the two loci start from the origin and breakaway at the origin itself.

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{3}$$

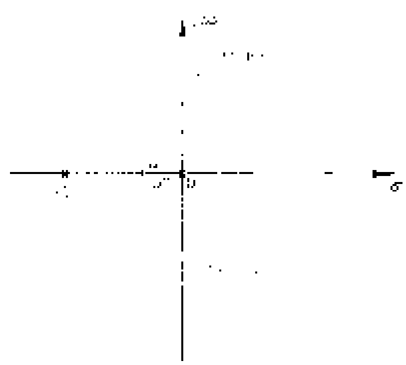
$$\theta_c = 60^0 \ \ k = 0$$

$$\theta_c = 180^0 \ \ k = 1$$

$$\theta_c = 300^0 \ \ k = 2$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + ...p_n) - (z_1 + z_2 + ...z_m)}{(n-m)} = \frac{0-1}{3} = -\frac{1}{3}$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s) = s^3 + s^2 + K$

$B(j\omega) = (j\omega)^3 + (j\omega)^2 + K = (K - \omega^2) - j\omega^3$

When imaginary-part is zero, then $\omega = 0 \Rightarrow s = 0$ and when real-part is zero, then $K = \omega^2 = 0$.

The root-locus does not cross the imaginary axis for any value of K>0.

Additional closed-loop poles

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles
1	-1.5	1.125	0.25±j0.82
2	-2.0	4	0.50±j1.32
3	-2.5	9.375	0.75±j1.78
4	-3.0	18	1.00±j2.23

Determine the gain corresponding to s=-1.5

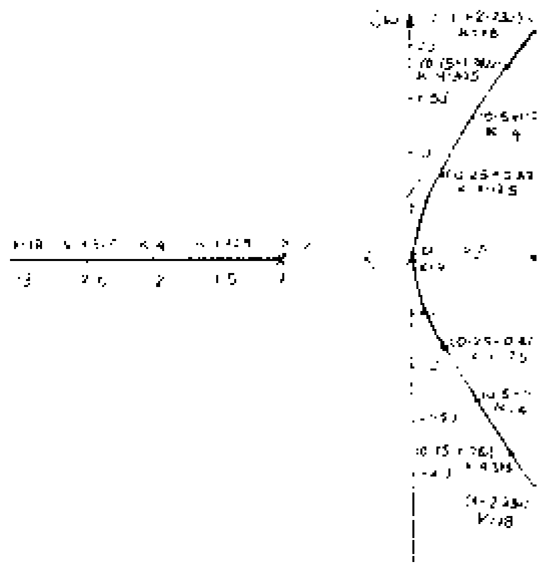
K=|-1.5-(-1)||-1.5-(0)||-1.5-0|= 1.125

$s^3 + s^2 + 1.125 = 0$

1	1	0	1.125
	-1.5	0.75	-1.125
1	-0.5	0.75	0

$(s^2 + 1.5s + 1.25) = 0$

$s_{2,3} = -0.25 \pm j0.82$



Problem-4: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \dfrac{K}{s^4 + 5s^3 + 8s^2 + 6s}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

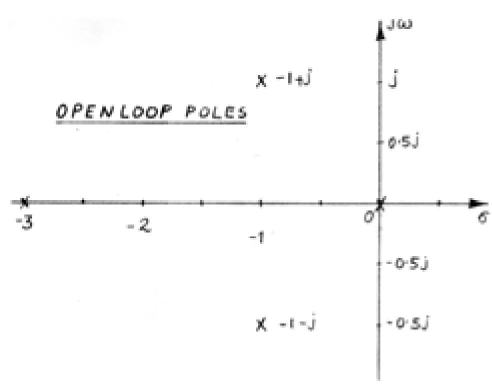
$s^4 + 5s^3 + 8s^2 + 6s = s(s^2 + 2s + 2)(s+3) = (s+1-j)(s+1+j)(s+3)s$

Number of open-loop poles n=4

Number of open-loop zeros m=0

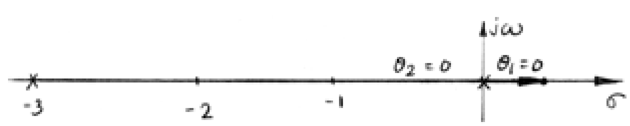
Open-loop poles: s=0 and s=-3, s=-1+j and s=-1-j

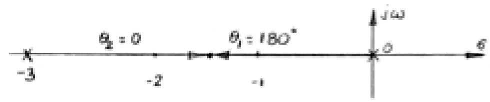
Step 2: Mark open-loop poles and zeros on the s-plane

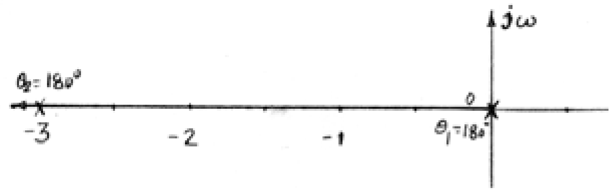


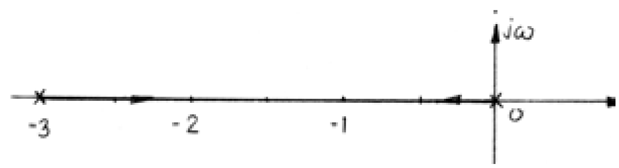
Step 3: Determine parts of the root-locus on the real axis

Test points on the positive real axis









Step 4: Determine breakaway and break-in point

Characteristic equation, $K = -(s^4 + 5s^3 + 8s^2 + 6s)$

$\frac{dK}{ds} = 0$

$\Rightarrow 4s^3 + 15s^2 + 16s + 6 = 0$

$\Rightarrow s^3 + 3.75s^2 + 4s + 1.5 = 0$

$f^{'}(s) = 3s^2 + 7.5s + 4$

This equation is solved using Newton-Raphson's method

$$s_{n+1} = s_n - \frac{f(s_n)}{f^{'}(s_n)}$$

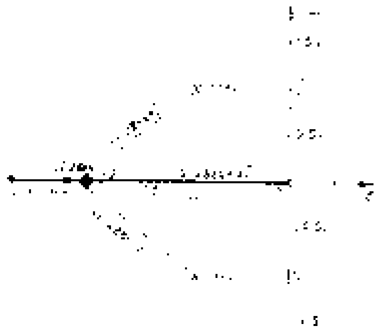
No.	s_n	$f(s_n)$	$f^{'}(s_n)$	s_{n+1}
1	-3.75	-13.5	18.0625	-3.0026
2	-3.0026	-3.7721	8.5273	-2.5602
3	-2.5602	-0.9421	4.4624	-2.3491
4	-2.3491	-0.1658	2.9364	-2.2926
5	-2.2926	-0.0103	2.5737	-2.2886
6	-2.2886	-5.03×10^{-5}		

Breakaway point as $\sigma_b = -2.3$

Gain at the breakaway point, $K = |-2.3-(-3)||-2.3-0||-2.3-(-1+j)||-2.3-(-1-j)| = 4.33$

1	5	8	6	K
	-2.2886	-6.2053	-4.1073	-4.3316

1	2.7114	1.7947	1.8926	0



Other closed-loop poles for K=4.3

1	2.7114	1.7947	1.893
	-2.2886	-0.9676	-1.893
1	0.4228	0.8270	0

$s_{3,4}$ =-0.2114±j0.8814

Step 5: Draw asymptotes of the root-locus
Angle of asymptotes:

$$\theta_c = \frac{180^0 + k360}{(n-m)} = \frac{180 \pm 360k}{4}$$

$$\theta_c = 45^0 \;\; k = 0$$

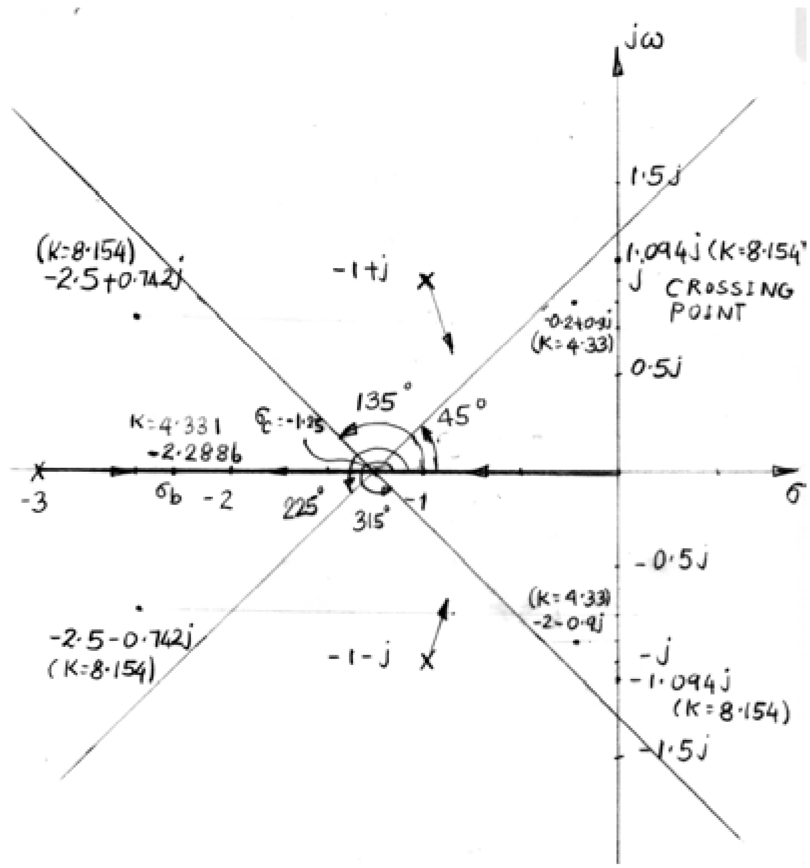
$$\theta_c = 135^0 \;\; k = 1$$

$$\theta_c = 225^0 \;\; k = 2$$

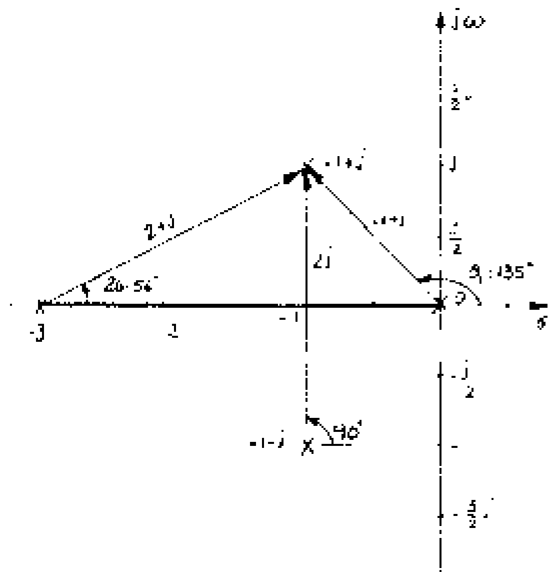
$$\theta_c = 315^0 \;\; k = 3$$

Centroid of asymptotes

$$\sigma_c = \frac{(p_1 + p_2 + \dots p_n) - (z_1 + z_2 + \dots z_m)}{(n-m)} = \frac{0 - 3 - 1 + j - 1 - j}{4} = -1.25$$



Steps 6: Determine angles of departure



$$\theta_d = 180^0 - (135^0 + 26.56^0 + 90^0) = -71.56^0 = 288.44^0$$

Step 7: As there are no complex open-loop zeros, angle of arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s) = s^4 + 5s^3 + 8s^2 + 6s + K$

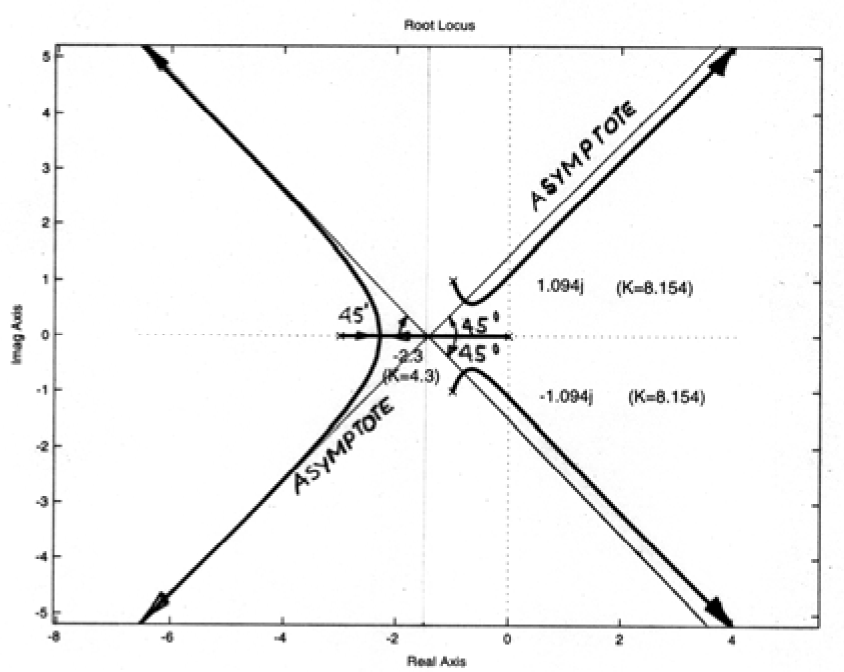
$B(j\omega) = (j\omega)^4 + 5(j\omega)^3 + 8(j\omega)^2 + 6j\omega + K = (\omega^4 - 8\omega^2 + K) + j(6\omega - 5\omega^3)$

When imaginary-part is zero, then $\omega = \pm\sqrt{\frac{6}{5}} \Rightarrow s = \pm j\sqrt{\frac{6}{5}}$ and when real-part is zero,

then $K = 8\times\left(\frac{6}{5}\right) - \left(\frac{6}{5}\right)^2 = 8.16$.

There are two closed-loop poles on the imaginary axis for any value of K>0.
Additional closed-loop poles

No.	S_1	S_2	$S_{3,4}$	K
1	-0.25	-2.9217	-0.9142±0.7969j	1.0742
2	-0.50	-2.8804	-0.8098±0.655j	1.5625
3	-0.75	-2.8593	-0.6953±0.5938j	1.7930
4	-1.0	-2.8393	-0.5804±0.6063j	2.0000
5	-1.25	-2.8055	-0.4722±0.6631j	2.3242
6	-1.75	-2.6562	-0.3763±0.7354j	2.8125
7	-2.0	-2.5214	-0.2393±0.8579j	4.0



Additional Information from Root-Locus Plot

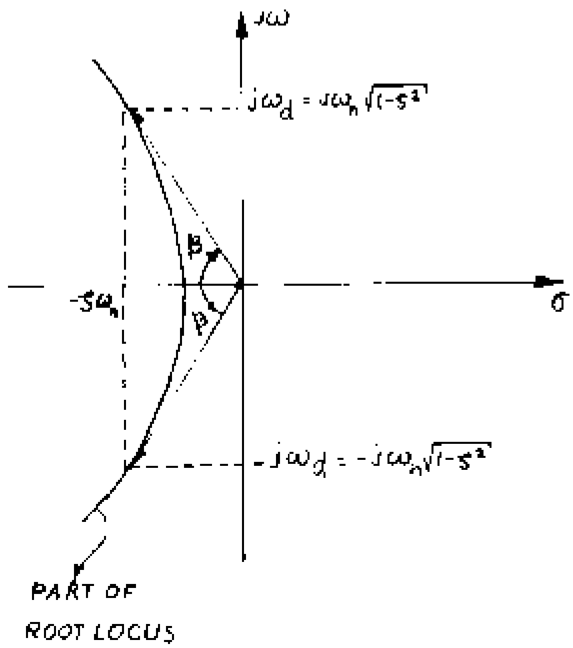
6. Gain Margin

$$GM = 20\log\frac{K_2}{K_1} \tag{10.69}$$

K_1 is the gain of a feedback system at some point on the root-locus

K_2 is the gain at which the system becomes unstable

7. Transient Characteristics



Where, $\beta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

8. Percentage overshoot

$$M_p = e^{-\pi/\tan\beta} \tag{10.70}$$

9. Settling time

$$t_s = \frac{4}{\zeta\omega_n} \tag{10.71}$$

10. Steady-state error is also related to K.

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) = \frac{K\left(s^2+10s+100\right)}{s^4+20s^3+100s^2+500s+1500}, H\left(s\right)=1$

(a) Determine the value of gain at which the system will be stable and as well have a maximum overshoot of 5%.

(b) What is the gain margin at this point?

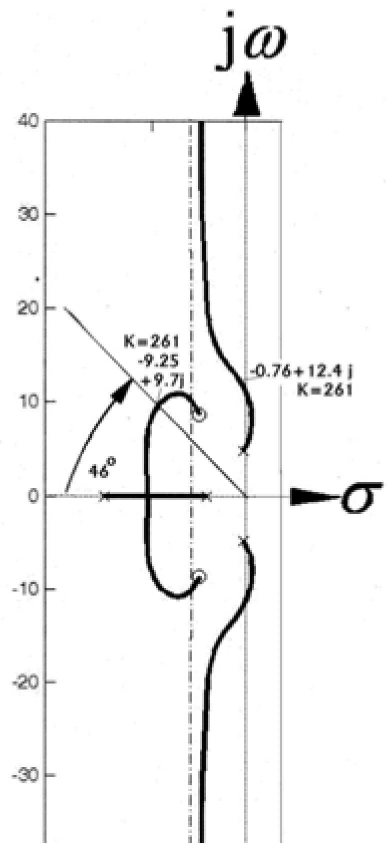
(c) What is the steady-state error for a unit step excitation at the above point?

Solution:

(b) $\tan\beta = \dfrac{-\pi}{\ln M_p} = 1.0487$

$\Rightarrow \beta = 46^0$

$$\zeta = \frac{1}{\sqrt{1+\tan^2\beta}} = 0.690 \qquad (10.72)$$



(b) $GM = 20\log\dfrac{192.2}{261} = -2.65dB$

(c) Position error

$$K_s = \lim_{s\to 0}\frac{K(s^2+10s+100)}{s^4+20s^3+100s^2+500s+1500} = \frac{100K}{1500}$$

Steady-state error,

$$S_e(\infty) = \frac{1}{1+K_s} = \frac{1}{1+100K/1500} = \frac{1500}{1500+100K}$$

$$S_e(\infty) = \frac{1500}{1500+100\times 261} = 5.4\%$$

Equation Chapter (Next) Section 1

a. Root Locus using MATLAB

Program 1: Draw the root locus for the following system $\dfrac{C(s)}{R(s)} = \dfrac{K}{(s+1)(s+2)(s+4)(s+5)}$

Solution:

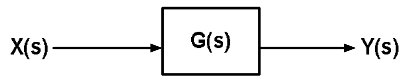
```
>> num=[01]
```

```
num=
0         1
>>q1=[1 1];
>> q2=[1 2];
>> q3=[1 3];
>> q4=[1 4];
>>den=conv(q1,q2);
>> den=conv(den,q3);
>> den=conv(den,q4);
den=
1         12        49        78        40
>>sys=tf(num,den)
Transfer function:
  1
--------------------
   s^4+12s^3+49s^2+78s+40
>>rlocus(sys)
```

11. Frequency Response Analysis

11.1. Frequency Response

This is defined as the steady-state response of a system due to a sinusoidal input.



Here,

$$G(s)=\frac{C(s)}{R(s)}=\frac{N(s)}{(s+a)(s+b)(s+c)...} \tag{11.1}$$

$$\Rightarrow C(s)=\frac{N(s)R(s)}{(s+a)(s+b)(s+c)...} \tag{11.2}$$

Let, $r(t)=A\sin\omega t$, then

$$R(s)=\frac{A\omega}{s^2+\omega^2} \tag{11.3}$$

Using eq (3) in eq (2),

$$\begin{aligned} C(s)&=\frac{N(s)}{(s+a)(s+b)(s+c)...}\left[\frac{A\omega}{s^2+\omega^2}\right] \\ \Rightarrow C(s)&=\frac{A_1}{s+a}+\frac{A_1}{s+b}+\frac{A_1}{s+c}+...+\frac{B_1}{s+j\omega}+\frac{B_2}{s-j\omega} \end{aligned} \tag{11.4}$$

In time domain, eq (5) becomes

$$c(t)=A_1e^{-at}+A_2e^{-bt}+A_3e^{-ct}+...+B_1e^{-j\omega t}+B_2e^{j\omega t} \tag{11.5}$$

The term with A_i terms are decaying components. So, they tend to zero as time tends to infinity. Then, eq (5) becomes

$$C_{ss}(t)=B_1e^{-j\omega t}+B_2e^{j\omega t} \tag{11.6}$$

Where,

$$\begin{aligned} B_1&=\left.\frac{A\omega G(s)}{s-j\omega}\right|_{s=-j\omega}=\frac{-A}{2j}\left|G(-j\omega)\right|e^{j\angle G(-j\omega)} \\ B_2&=\left.\frac{A\omega G(s)}{s+j\omega}\right|_{s=j\omega}=\frac{A}{2j}\left|G(j\omega)\right|e^{j\angle G(j\omega)} \end{aligned} \tag{11.7}$$

Since, $\left|G(j\omega)\right|=\left|G(-j\omega)\right|$ and $\angle G(-j\omega)=\angle G(j\omega)=\phi$

$$c(t) = \frac{-A}{2j}\left|G(j\omega)\right|e^{-j(\omega t+\phi)} + \frac{A}{2j}\left|G(j\omega)\right|e^{j(\omega t+\phi)} \tag{11.8}$$

$$\Rightarrow c(t) = -A\left|G(j\omega)\right|e^{-j\omega t}\left[\frac{e^{j\phi} - e^{-j\phi}}{2j}\right] \tag{11.9}$$

$$\Rightarrow c(t) = A\left|G(j\omega)\right|\sin(\omega t+\phi) \tag{11.10}$$

$$\Rightarrow c(t) = B(\omega)\sin(\omega t+\phi) \tag{11.11}$$

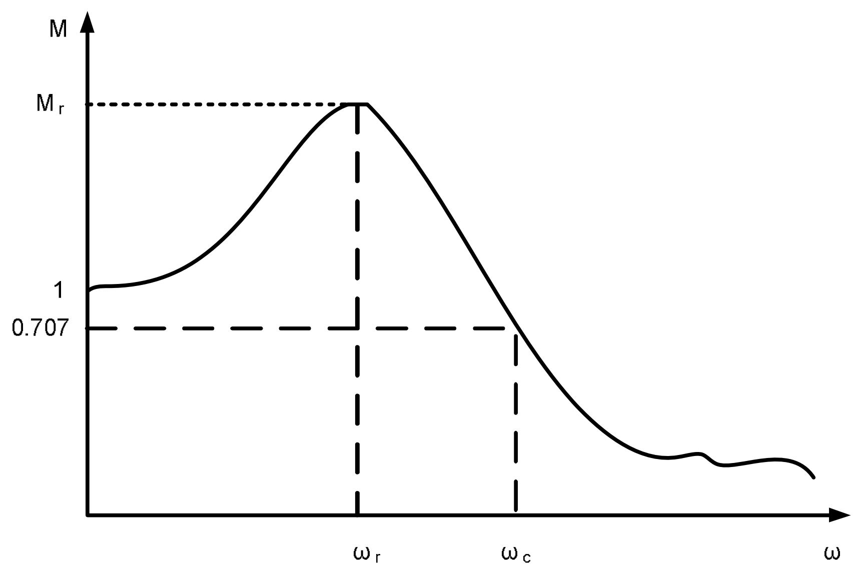
Where, $B(\omega) = A\left|G(j\omega)\right|$

Therefore, the steady-state response of the system for a sinusoidal input of magnitude A and frequency ω is a sinusoidal output with a magnitude $B(\omega)$, frequency ω and phase shift ϕ.

The following plots are used in frequency response.

- Polar plot
- Bode plot
- Magnitude versus phase angle plot

11.2. Definition of frequency domain specifications



(i) Resonant peak (M_r): Maximum value of $M(j\omega)$ when ω is varied from 0 to ∞.

(ii) Resonant frequency (ω_r): The frequency at which M_r occurs

(iii) Cut-off frequency (ω_c): The frequency at which $M(j\omega)$ has a value $\frac{1}{\sqrt{2}}$. It is the frequency at which the magnitude is 3dB below its zero frequency value

(iv) Band-width (ω_b): It is the range of frequencies in which the magnitude of a closed-loop system is $\frac{1}{\sqrt{2}}$ times of M_r

(v) Phase cross-over frequency: The frequency at which phase plot crosses -180^0

(vi) Gain margin (GM): It is the increase in open-loop gain in dB required to drive the closed-loop system to the verge of instability

(vii) Gain cross-over frequency: The frequency at which gain or magnitude plot crosses 0dB line

(viii) Phase margin (PM): It is the increase in open-loop phase shift in degree required to drive the closed-loop system to the verge of instability

11.3. Correlation between time and frequency response

For a second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{11.12}$$

Putting $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}$$
$$\Rightarrow \frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\left(\frac{\omega}{\omega_n}\right)} \tag{11.13}$$

Let, $u = \frac{\omega}{\omega_n}$, then

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - u^2\right) + j2\zeta u} \tag{11.14}$$

Now,

$$M(j\omega) = \left|M(j\omega)\right| \angle M(j\omega) \tag{11.15}$$

Where,

$$\left|M(j\omega)\right| = \frac{1}{\sqrt{\left(1-u^2\right)^2 + \left(2\zeta u\right)^2}}$$
$$\theta = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) \tag{11.16}$$

Now,

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{11.17}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \tag{11.18}$$

$$\omega_b = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{11.19}$$

$$PM = -180^0 + \varphi \tag{11.20}$$

Where, $\varphi = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{4\zeta^2 + 1} - 2\zeta^2}}$

11.4. Advantages

- Good accuracy
- Possible to test in lab
- Can be used to obtain transfer function that is not possible with analytical techniques
- Easy to design open-loop transfer function from closed-loop performance in frequency domain
- It is very easy to visualize the effect of disturbance and parameter variations.

11.5. Disadvantages

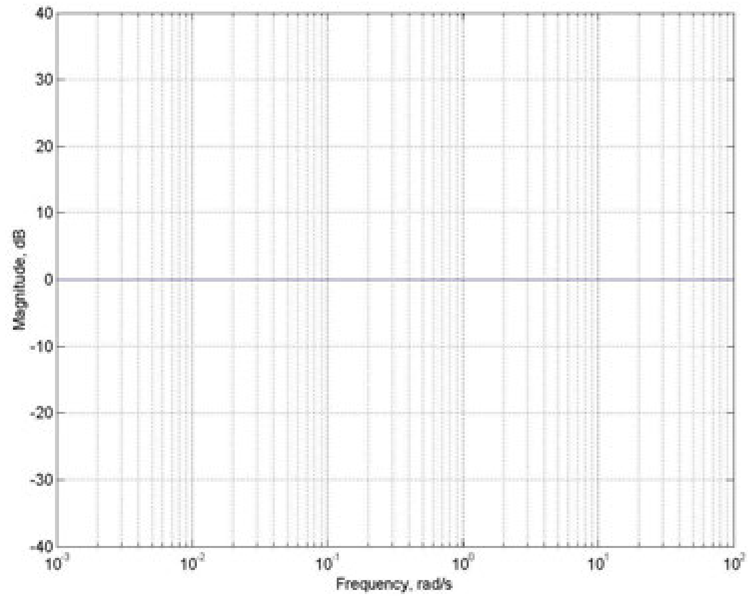
- Applied only to linear systems
- Frequency response for existing system is possible to obtain if the time constant is up to few minutes
- Time consuming procedure
- Old and back dated method

Equation Chapter 12 Section 1

12. Bode Plots

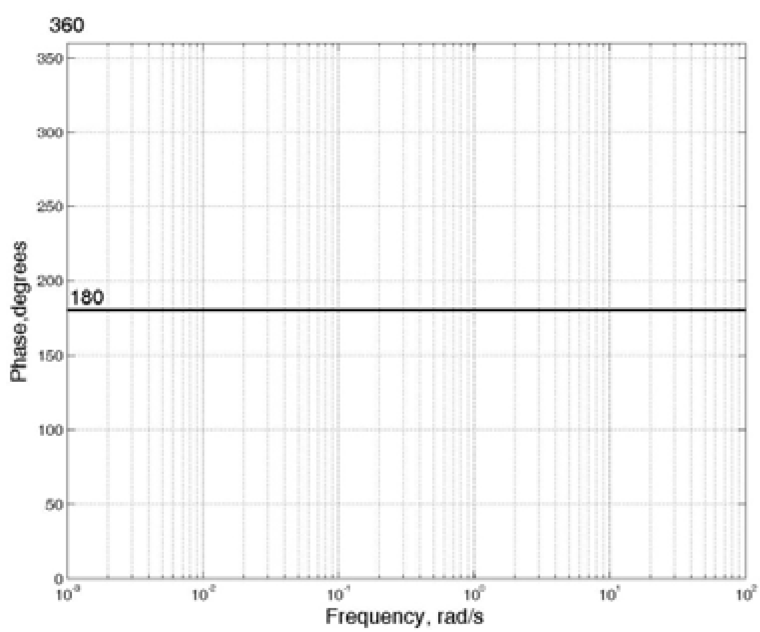
12.1. Magnitude plot and phase plot on a semi-log paper

Magnitude plot on a semi-log paper



$M = 20\log|G(j\omega)H(j\omega)|\,\text{dB}$

Phase plot on a semi-log paper



12.2. Magnitude versus phase Bode plot Nichols plot

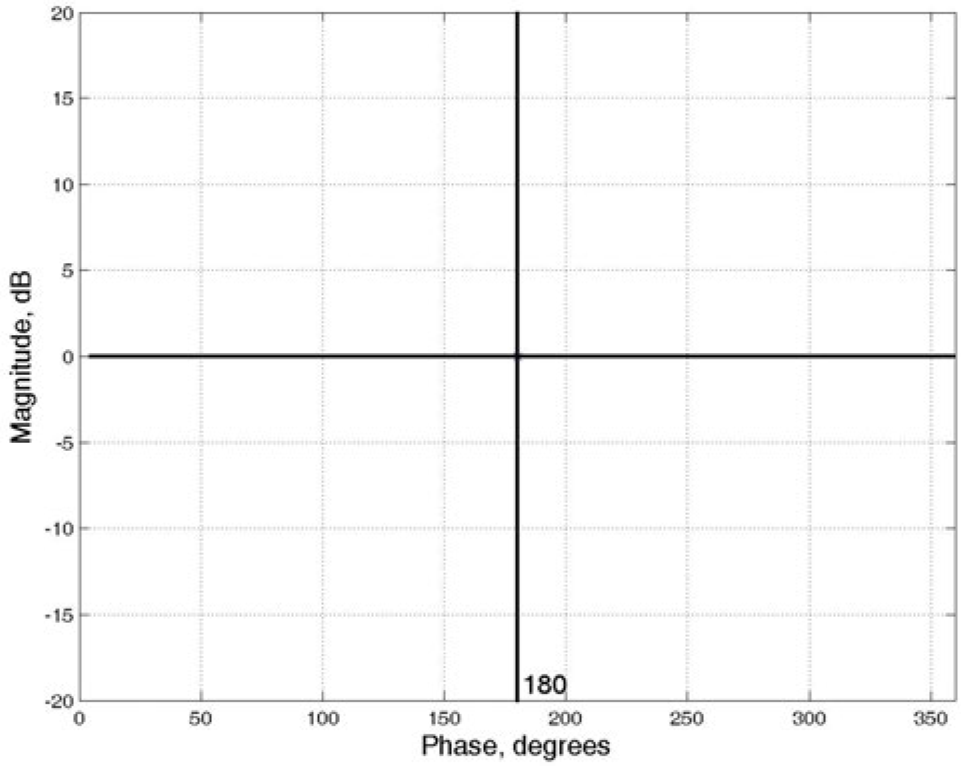


Table 12.1 Basic frequency response factors

No	Laplace term	Frequency response	Type of factor
1	K	K	Constant
2	s	$j\omega$	Derivative factor
3	1/s	$1/j\omega$	Integral factor
4	$\tau s+1$	$(1+j\omega\tau)$	First order derivative factor
5	$1/(\tau s+1)$	$1/(1+j\omega\tau)$	First order integral factor
6	$s^2+2\varsigma\omega_n s+\omega_n^2$	$\omega_n^2-\omega^2+j2\varsigma\omega_n\omega$	Second order derivative factor
7	$\frac{1}{s^2+2\varsigma\omega_n s+\omega_n^2}$	$\frac{1}{\omega_n^2-\omega^2+j2\varsigma\omega_n\omega}$	Second order integral factor

12.3. Derivative factor: magnitude

$$M = 20\log|j\omega| = 20\log\omega \text{ dB} \tag{12.1}$$

$$\angle j\omega = 90^0 \tag{12.2}$$

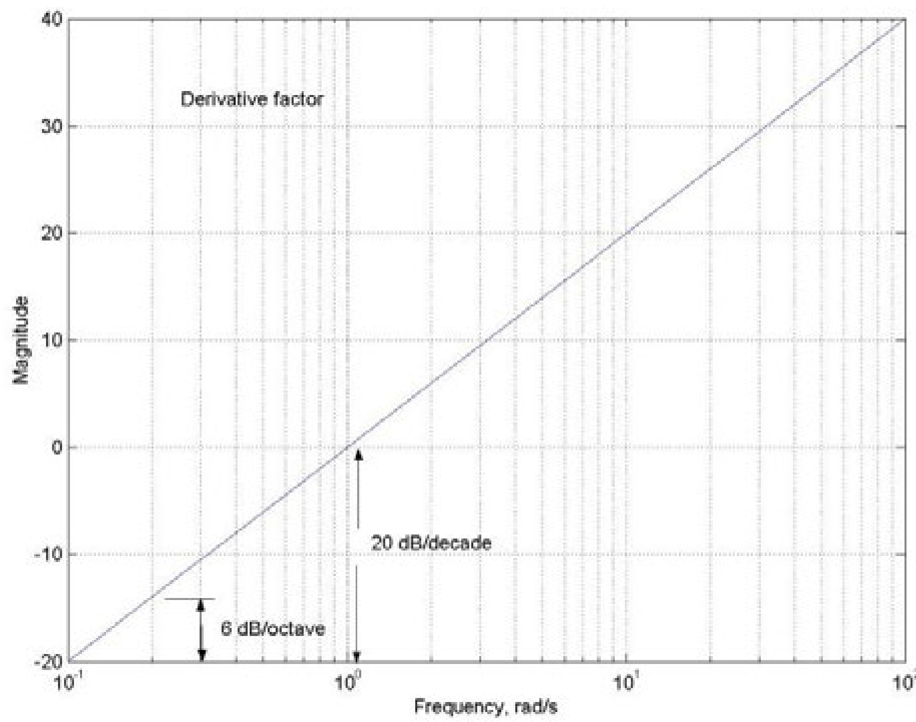
$$\Delta M = 20\log\omega_2 - 20\log\omega_1 = 20\log\frac{\omega_2}{\omega_1} \text{ dB/decade} \tag{12.3}$$

$$\Delta M = 20\log 10 = 20 \text{ dB/decade} \tag{12.4}$$

$$\Delta M = 20\log 2 \approx 6 \text{ dB/octave} \tag{12.5}$$

Table 12.2 Magnitude variation of a derivative factor for various multiples of the initial frequency

$\frac{\omega_2}{\omega_1}$	1	2	3	4	5	6	7	8	9	10
ΔM dB	0	6	10	12	14	16	17	18	19	20



12.4. Derivative Factor: (phase)

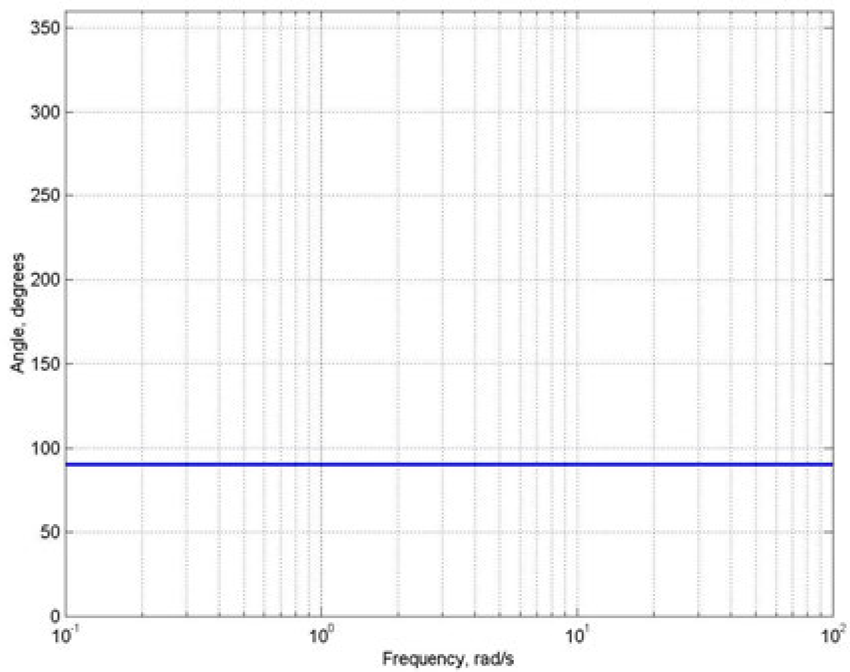


Table 15.3Derivative factor

	Frequency, rad/s				
	0.1	1	10	30	100
Magnitude, dB	-20	0	20	30	40
Phase, degrees	90	90	90	90	90

12.5. Integral factor: magnitude

$$M = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB} \tag{12.6}$$

$$\angle j\omega = 270^{0} \tag{12.7}$$

$$\Delta M = -20\log\omega_2 + 20\log\omega_1 = -20\log\frac{\omega_2}{\omega_1} \text{ dB/decade} \tag{12.8}$$

$$\Delta M = -20\log 10 = -20 \text{ dB/decade} \tag{12.9}$$

$$\Delta M = 20\log 2 \approx -6 \text{ dB/octave} \tag{12.10}$$

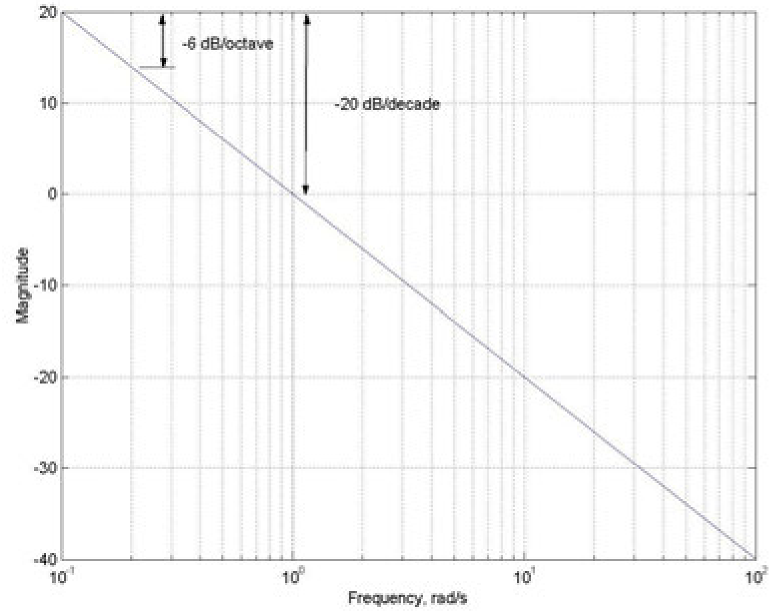


Table 12.4 Magnitude variation of an integral factor for various multiples of the initial frequency

$\frac{\omega_2}{\omega_1}$	1	2	3	4	5	6	7	8	9	10
ΔM , dB	0	-6	-10	-12	-14	-16	-17	-18	-19	-20

12.6. Integral factor: phase

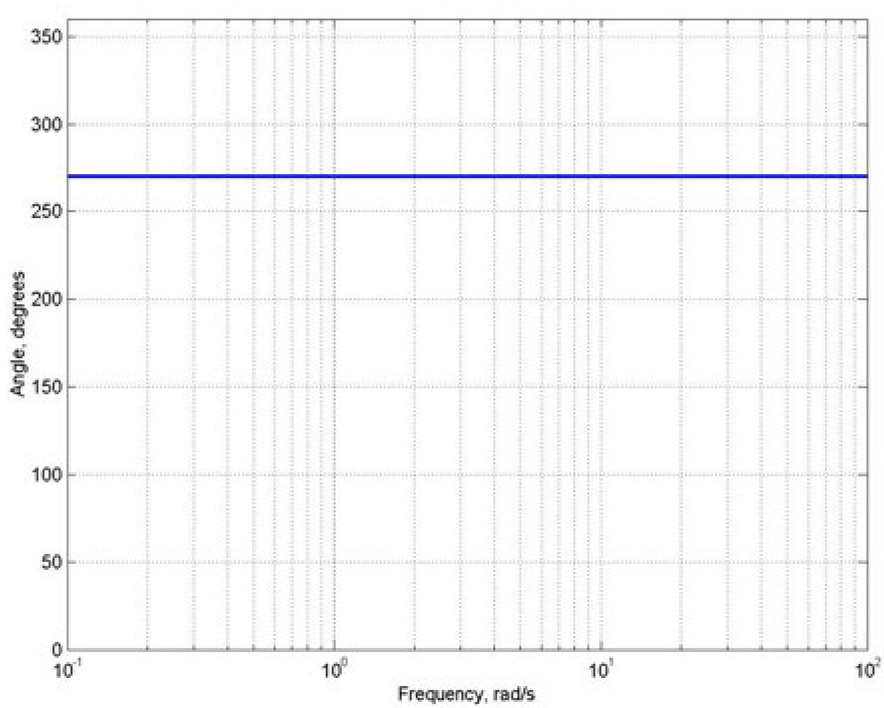


Table 12.5 Bode magnitude and phase of an integral factor

	Frequency, rad/s				
	0.1	1	10	20	100
Magnitude, dB	20	0	-20	-26	-40
Phase, degrees	270	270	270	270	270

12.7. First-order derivative factor: magnitude

$$M = 20\log\left|1 + j\omega\tau\right| = 20\log(\sqrt{1+\left[\omega\tau\right]^2} \text{ dB} \quad (12.11)$$

For $\omega \ll \omega_c$, $M \approx 0$ dB

For $\omega \gg \omega_c$,

$$M \approx 20\log\frac{\omega}{\omega_c} \text{ dB} \quad (12.12)$$

Here, $\omega_c = 1/\tau$ = corner frequency

For $\omega > \omega_c$

$$\Delta M = 20\log\omega_2 - 20\log\omega_1 = 20\log\frac{\omega_2}{\omega_1} \quad (12.13)$$

$$\Delta M = 20\log 10 = 20 \text{ dB/decade} \quad (12.14)$$

$$\Delta M = 20\log 2 \approx 6 \text{ dB/octave} \quad (12.15)$$

Table 12.6 Magnitude variation of a first-order derivative factor for various multiples of the corner frequency

$\frac{\omega}{\omega_c}$	1	2	3	4	5	6	7	8	9	10
ΔM, dB	0	6	10	12	14	16	17	18	19	20

12.8. First-order derivative factor: phase

$$\theta = \angle 1 + j\omega\tau = \arctan\left(\omega\tau\right) \quad (12.16)$$

$$\begin{aligned} &\theta \approx 0 && ; w < \frac{w_c}{10} \\ &\theta = 45^0\left(1 + \log\frac{\omega}{\omega_c}\right) && ; \frac{w_c}{10} < w < 10w_c \\ &\theta \approx 90 && ; w > 10w_c \end{aligned} \quad (12.17)$$

Table 12.7 Phase angles of a first-order derivative factor around the corner frequency

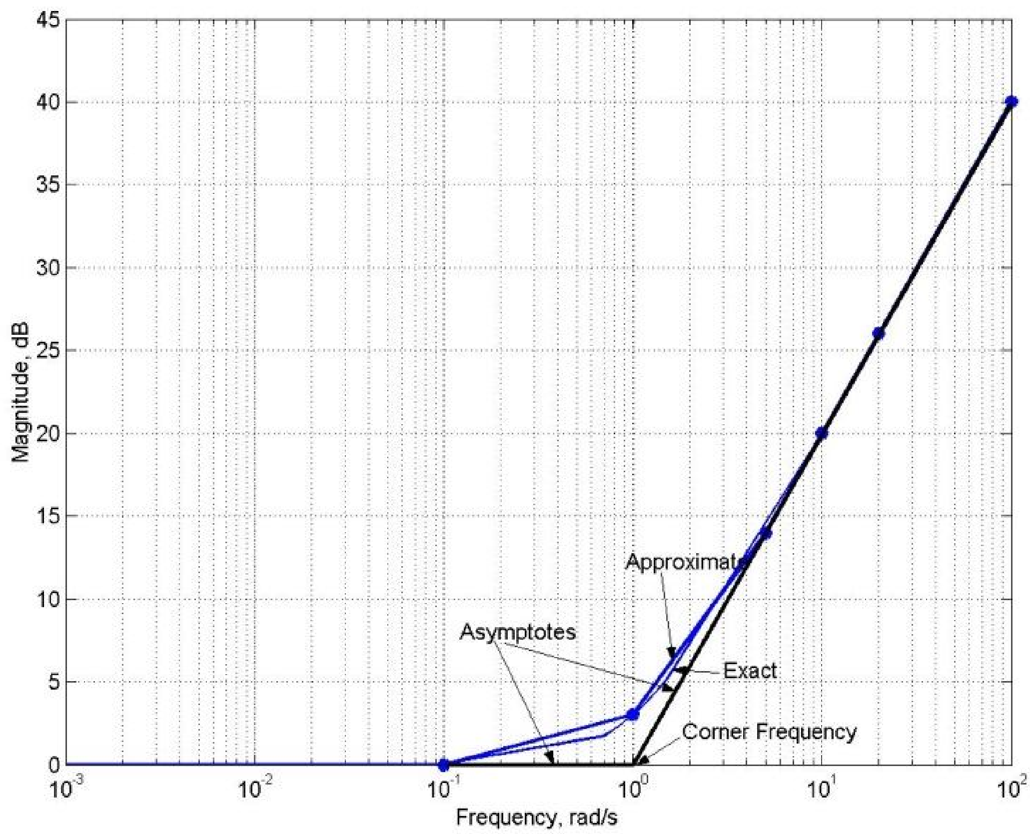
$\frac{\omega}{\omega_c}$	1	2	3	4	5	6	7	8	9	10
θ , deg	45	59	66	72	76	80	83	86	88	90
$\frac{\omega}{\omega_c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
θ, deg	0	2	4	7	10	14	18	24	31	45

12.9. First-order derivative factor
For $\tau = 1$

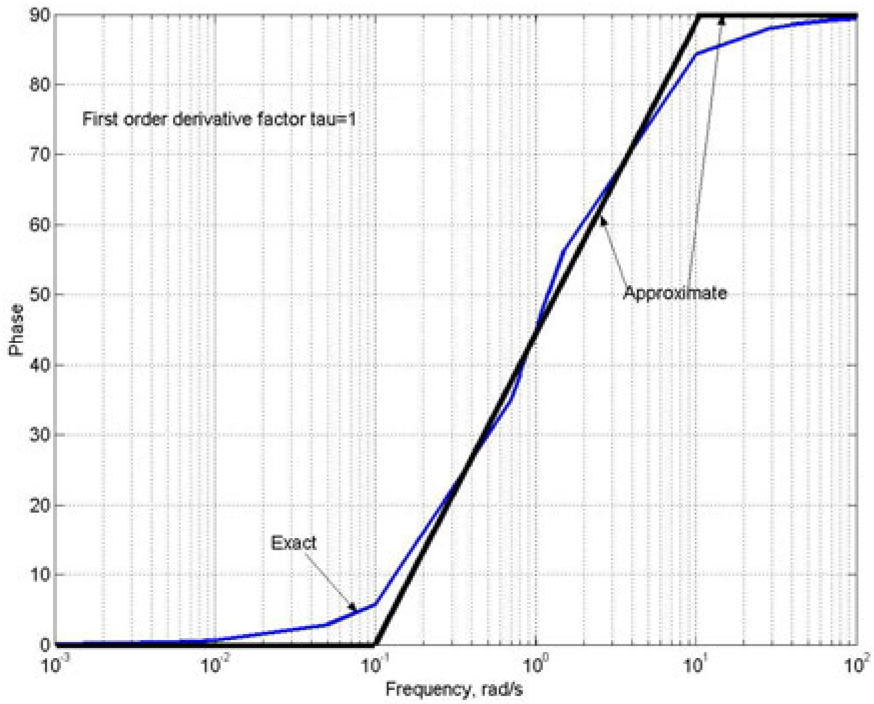
Table 15.8 Bode magnitude and phase

	Frequency, rad/s					
	0.1	1	5	10	20	100
Magnitude, dB	0	3	14	20	26	40
Phase, degrees	0	45	76	90	90	90

First-order derivative factor: magnitude (3 dB correction at the corner frequency)



First-order derivative Factor: phase



12.10. First-order integral factor: magnitude

$$M = 20\log\left|\frac{1}{1+j\omega\tau}\right| = 20\log\left(\frac{1}{\sqrt{1+[\omega\tau]^2}}\right)\text{dB} \tag{12.18}$$

$$\begin{aligned} M &\approx 0, & w << w_c \\ M &\approx -20\log\frac{\omega}{\omega_c}dB, & w >> w_c \end{aligned} \tag{12.19}$$

$$\Delta M = -20\log\omega_2 + 20\log\omega_1 = -20\log\frac{\omega_2}{\omega_1}\text{dB/decade} \tag{12.20}$$

$$\Delta M = -20\log 2 \approx -6\text{ dB/octave} \tag{12.21}$$

Table 12.9 Magnitude variation of a first-order integral factor for various multiples of the corner frequency

$\frac{\omega}{\omega_c}$	1	2	3	4	5	6	7	8	9	10
ΔM, dB	0	-6	-10	-12	-14	-16	-17	-18	-19	-20

Table 12.10 Phase angles of a first-order integral factor around the corner frequency

$\frac{\omega}{\omega_c}$	1	2	3	4	5	6	7	8	9	10
θ deg	315	301	294	288	284	280	277	274	272	270
$\frac{\omega}{\omega_c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
θ deg	360	358	356	353	350	346	342	336	329	315

First-order integral factor: phase

$\theta=360$, $\omega<\omega_c/10$

$$\theta = 360 - 45^0\left(1+\log\frac{\omega}{\omega_c}\right),\ \omega_c/10<\omega<10\ \omega_c$$

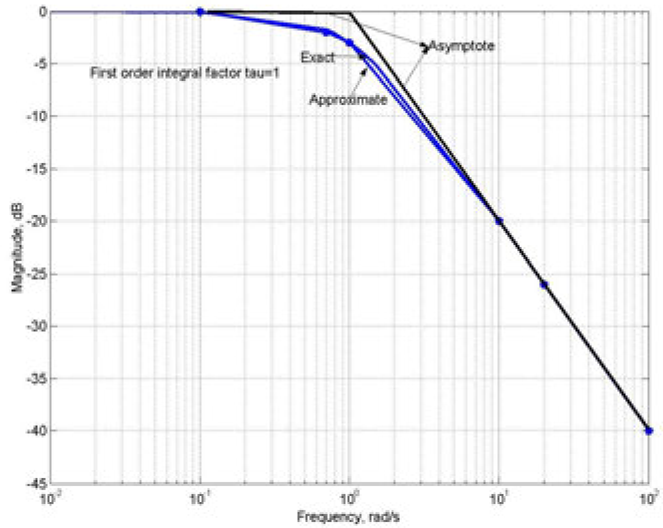
$$\theta = 360 - 45^0\left(1+\log\frac{\omega}{\omega_c}\right)$$

$\theta=270$, $\omega>10\ \omega_c$

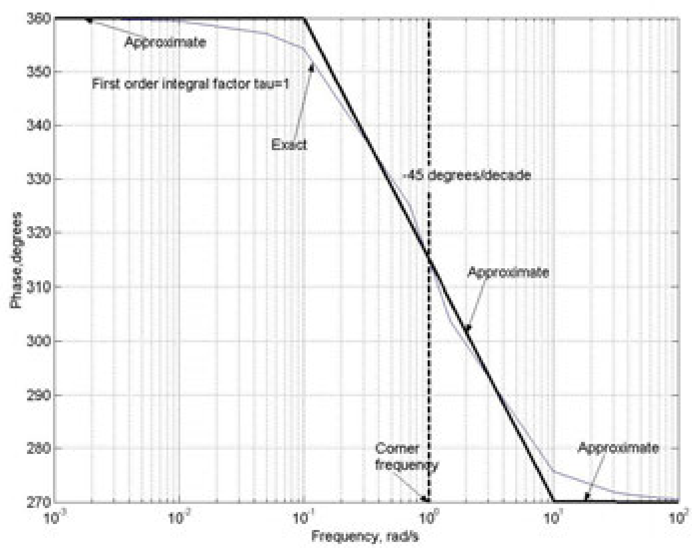
Table 12.11 Bode magnitude and phase of a first-order integral factor

	Frequency, rad/s							
	0.01	0.1	0.7	1	7	10	20	100
Magnitude, dB	0	0	-2	-3	-17	-20	-26	-40
Phase, degrees	360	360	322	315	277	270	270	270

First-order integral factor: magnitude



First-order integral factor: phase



12.12. Second-order derivative factor: magnitude

$$M = 20\log|\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n|$$

$$= 20\log\left(\omega_n^2\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right) \quad (12.22)$$

$$\mathbf{M \approx 40logw_n, w << w_n}$$
$$\mathbf{M = 20log\ (2zw_n{}^2), w = w_n} \quad (12.23)$$
$$\mathbf{M = 40\ logw, w >> w_n}$$

For $w >> w_n$

$$\Delta M = 40\log\omega_2 - 40\log\omega_1 = 40\log\frac{\omega_2}{\omega_1}\ \text{dB/decade} \tag{12.24}$$

$$\Delta M = 40\log 10 = 40\ \text{dB/decade} \tag{12.25}$$

$$\Delta M = 40\log 2 \approx 12\ \text{dB/octave} \tag{12.26}$$

Magnitude variation of a second-order derivative factor for various multiples of the resonant frequency

$\frac{\omega}{\omega_n}$	1	2	3	4	5	6	7	8	9	10
ΔM dB	0	12	20	24	28	32	34	36	38	40

Second-order derivative factor: phase

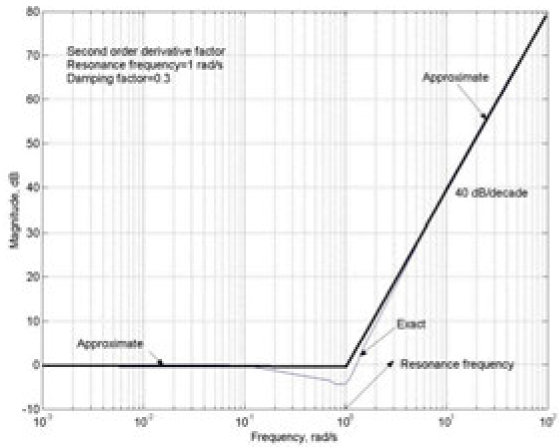
$$\theta = \angle|\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n| = \arctan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) \tag{12.27}$$

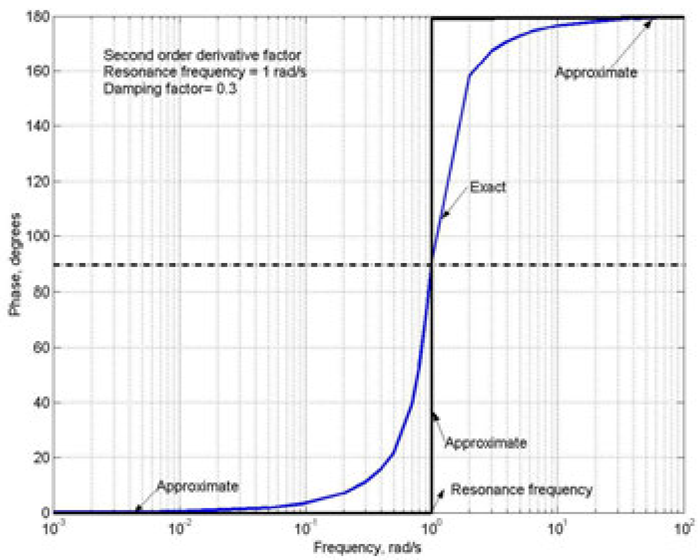
$$\begin{aligned} \theta &= 0^0, & w &< \frac{w_n}{10} \\ \theta &= 90^0, & w &= w_n \\ \theta &= 180^0, & w &> 10w_n \end{aligned} \tag{12.28}$$

Bode magnitude and phase

ω_n=1 rad/s, ζ=0.3

Frequency, rad/s	0.01	0.1	0.7	1	3	10	100
Magnitude, dB	0	0	-4	-4	18	40	80
Phase, degrees	0	0	39	90	167	180	180





Second-order integral factor

$$M = 20\log\left|\frac{1}{\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n}\right| dB = 20\log\left(\frac{1}{\omega_n^2\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}\right) dB \quad (12.29)$$

$M \approx -40\log\omega_n$, $\omega << \omega_n$

$M = -20\log(2\zeta\omega_n^2)$, $\omega = \omega_n$

$M = -40\log\omega$, $\omega >> \omega_n$

$$\Delta M = -40\log\omega_2 + 40\log\omega_1 = -40\log\frac{\omega_2}{\omega_1} dB/decade \quad (12.30)$$

$$\Delta M = -40\log 10 dB = -40 dB \qquad (12.31)$$

Magnitude variation of a second-order integral factor for various multiples of the resonant frequency

$\frac{\omega}{\omega_n}$	1	2	3	4	5	6	7	8	9	10
ΔM, dB	0	-12	-20	-24	-28	-32	-34	-36	-38	-40

$$\theta = \angle\left|\frac{1}{\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n}\right| = 360 - \arctan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) \qquad (12.32)$$

$\theta=0,\ \omega<\omega_n$

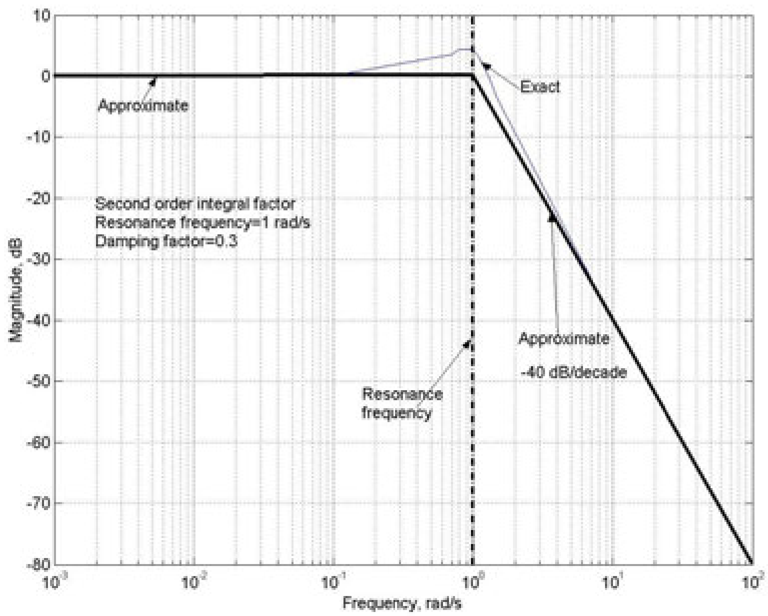
$\theta=270^0,\ \omega=\omega_n$

$\theta=180^0,\ \omega>\omega_n$

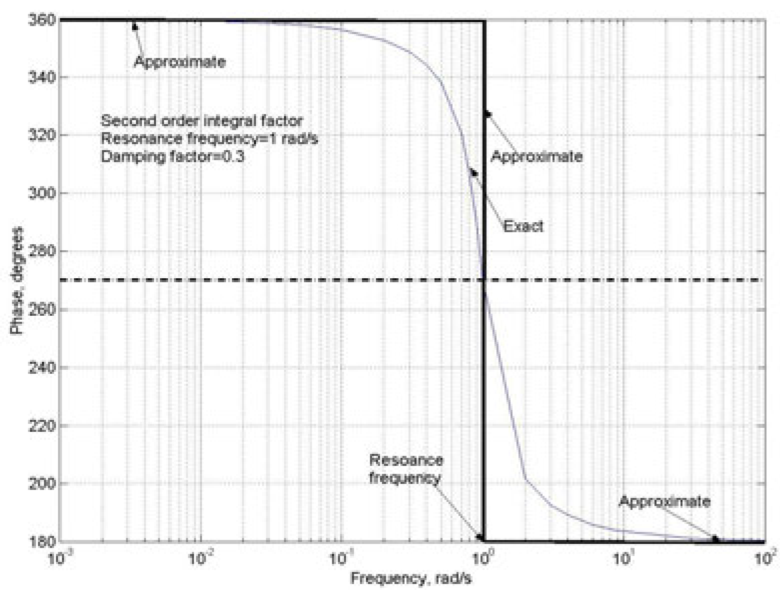
Bode magnitude and phase

Frequency, rad/s	0.01	0.1	0.7	1	3	10	100
Magnitude, dB	0	0	4	4	-18	-40	-80
Phase, degrees	360	360	321	270	193	180	180

Magnitude plot



Phase plot



Example 14.1

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s) = \frac{1}{s(s+1)}$$

Solution

Applying $s = j\omega$,

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

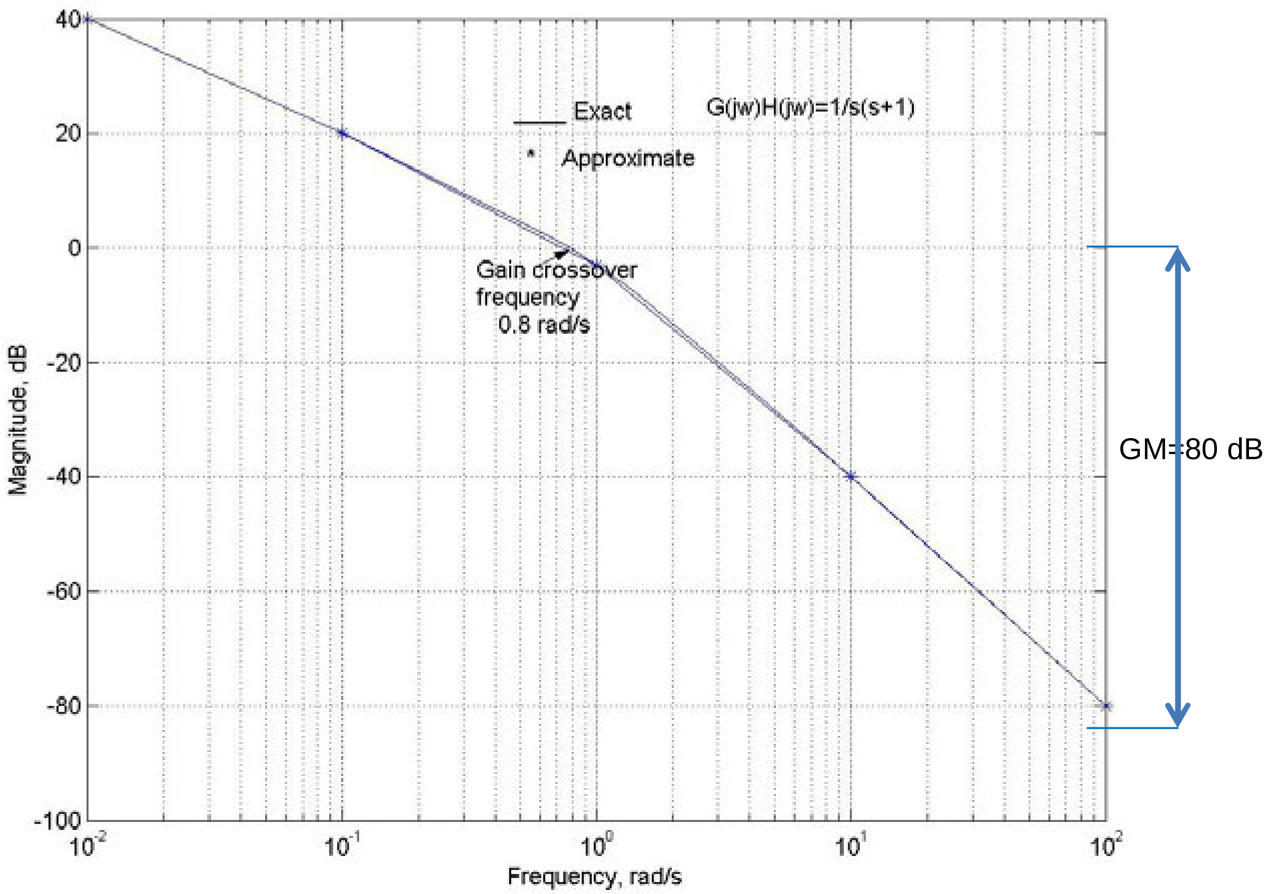
The above frequency response function has two factors: (1) Integral factor and (2) First order integral factor with a corner frequency of 1 rad/s

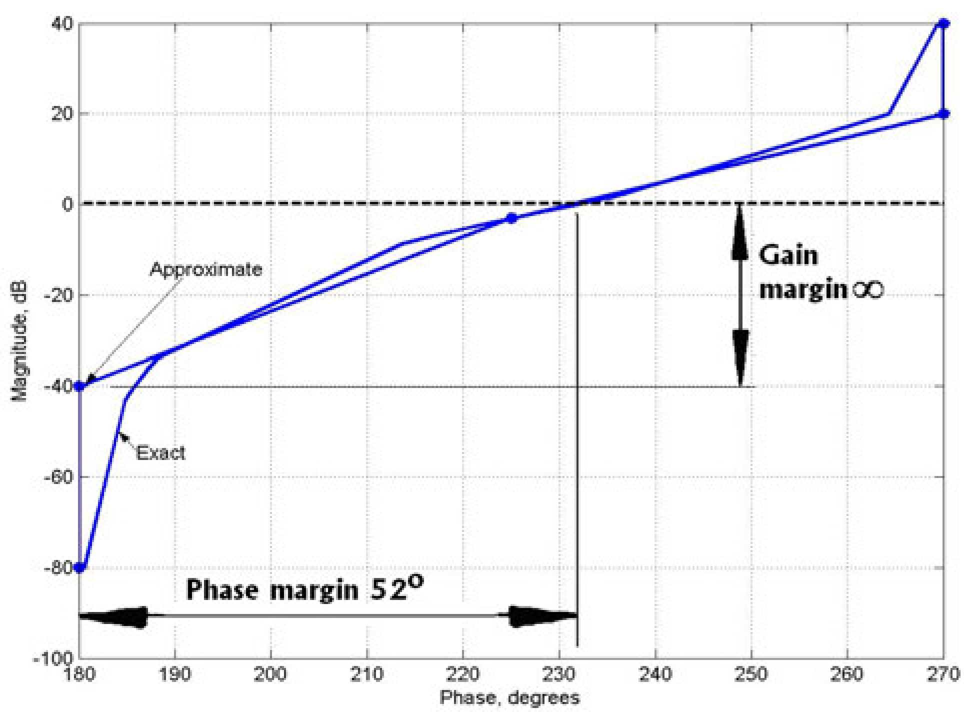
Bode magnitude of the transfer function

	Frequency, radians/s				
	0.01	0.1	1	10	100
$20\log\frac{1}{j\omega}$ dB	40	20	0	-20	-40
$20\log\frac{1}{j\omega+1}$ dB	0	0	-3	-20	-40
Magnitude, dB	40	20	-3	-40	-80

ω_p = 100 rad/s

	Frequency, rad/s				
	0.01	0.1	1	10	100
$\angle\frac{1}{j\omega}$ degrees	270	270	270	270	270
$\angle\frac{1}{j\omega+1}$ degrees	360	360	315	270	270
Bode phase, degrees	270	270	225	180	180


Exact
G(jw)H(jw)=1/s(s+1)
* Approximate
Gain crossover frequency 0.8 rad/s
GM=80 dB
Magnitude, dB
Frequency, rad/s
40
20
0
-20
-40
-60
-80
-100
10^-2
10^-1
10^0
10^1
10^2


Approximate
Exact
Gain margin ∞
Phase margin 52°
Magnitude, dB
Phase, degrees
40
20
0
-20
-40
-60
-80
-100
180
190
200
210
220
230
240
250
260
270

Example 14.2
Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s) = \frac{1}{s(s+2)\left(s+4)\right)}$$

Solution

$$G(j\omega)H(j\omega) = \frac{1}{8j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{j\omega}{4}+1\right)}$$

The corner frequencies corresponding to first order integral factors are 2 rad/s and 4 rad/s. Minimum frequency is chosen as 0.01 rad/s and maximum frequency 100 rad/s.

Table 14.1 Computation of Bode magnitude using asymptotic properties of the integral first-order term $\tau = \frac{1}{2}$

	x1	x2		x1	x10		x2	x1		x1	x2		x1	x10
Frequency, rad/s	2	4		2	20		20	10		20	40		10	100
Magnitude, dB	0	-6		0	-20		-20	-14		-20	-26		-14	-34

Table 14.2 Computation of Bode magnitude using asymptotic properties of the integral first-order term $\tau = \frac{1}{4}$

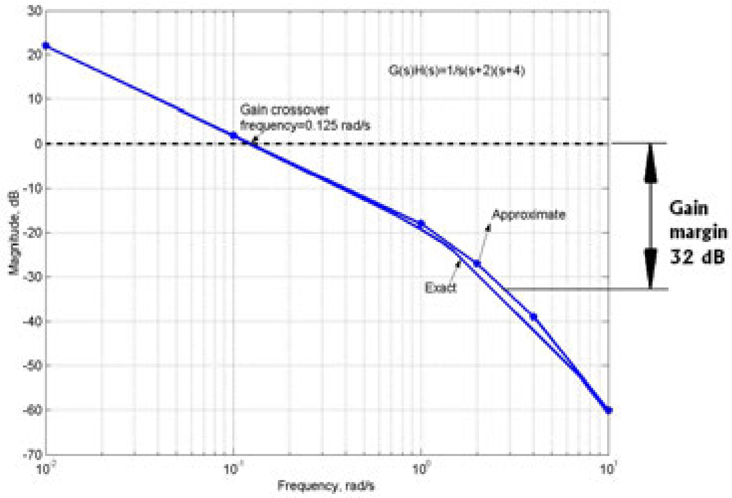
	x1	x10		x2	x1		x2	x1		x1	x10
Frequency, rad/s	4	40		40	20		20	10		10	100
Magnitude, dB	0	-20		-20	-14		-14	-8		-8	-28

Table 12.3 Bode magnitude

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$20\log\frac{1}{8}$	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18
$20\log\frac{1}{j\omega}$	40	20	14	8	0	-6	-12	-20	-26	-32	-40
$20\log\frac{1}{\frac{j\omega}{2}+1}$	0	0	0	0	-1	-3	-6	-14	-20	-26	-34
$20\log\frac{1}{\frac{j\omega}{4}+1}$	0	0	0	0	0	-1	-3	-8	-14	-20	-28
Bode magnitude,	22	2	-4	-10	-18	-28	-39	-60	-78	-96	-120

dB											

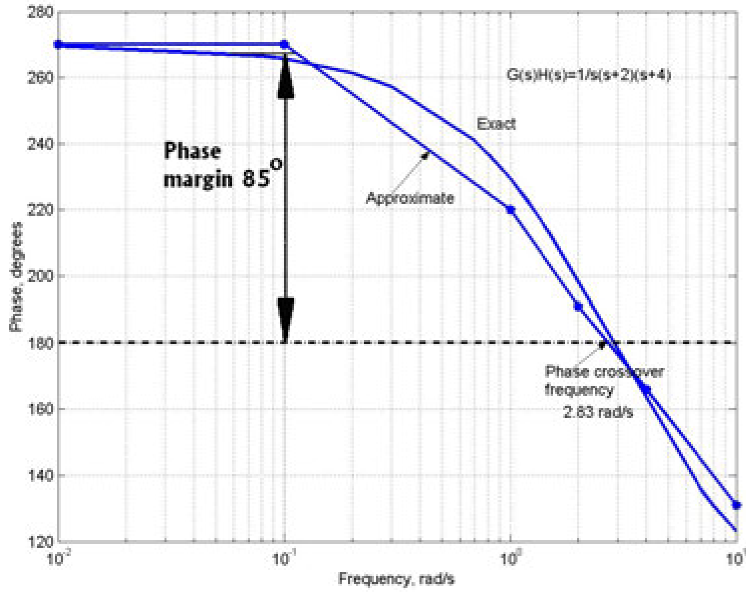
Bode magnitude



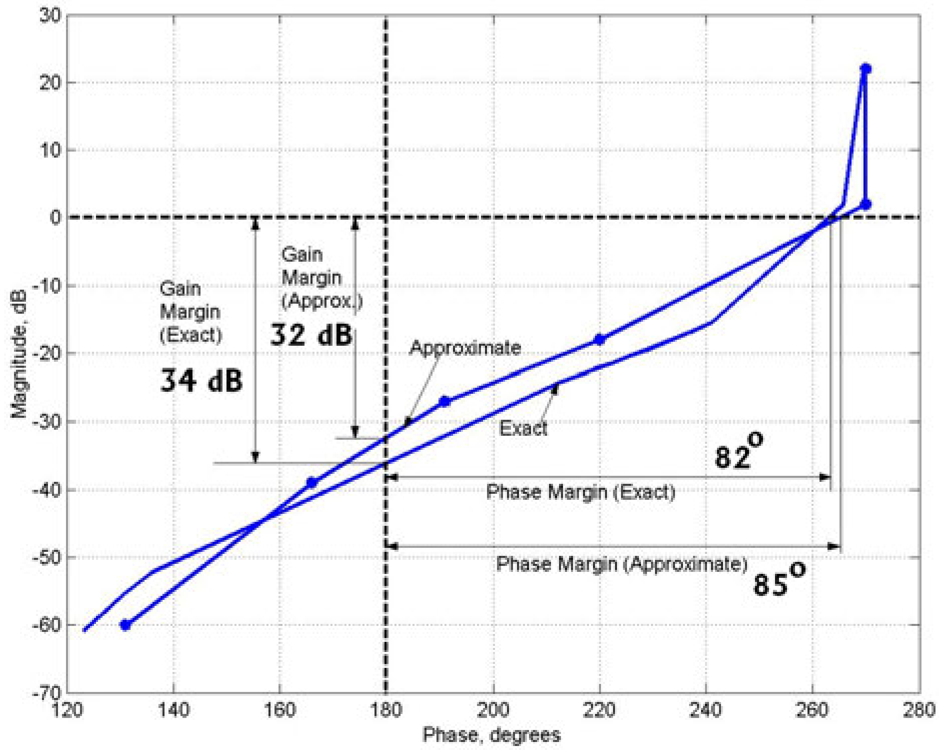
Bode phase

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$\angle\frac{1}{8}$	0	0	0	0	0	0	0	0	0	0	0
$\angle\frac{1}{j\omega}$	270	270	270	270	270	270	270	270	270	270	270
$\angle\frac{1}{\frac{j\omega}{2}+1}$	360	360	360	346	328	315	301	284	270	270	270
$\angle\frac{1}{\frac{j\omega}{4}+1}$	360	360	360	360	342	326	315	297	285	270	270
Phase degrees	270	270	270	256	220	191	166	131	105	90	90

Phase plot


G(s)H(s)=1/s(s+2)(s+4)
Exact
Approximate
Phase
margin 85°
Phase crossover
frequency
2.83 rad/s
Phase, degrees
Frequency, rad/s

Bode plot



Example 12.1
Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s) = \frac{1}{s^2(s+1)}$$

Solution

$$G(j\omega)H(j\omega) = \frac{1}{(j\omega)(j\omega)(j\omega+1)}$$

There are two integral factors and an integral first-order term with a corner frequency of 1 rad/s
Bode magnitude

	Frequency, rad/s				
	0.01	0.1	1	10	100
$20\log\frac{1}{j\omega}$ dB	40	20	0	-20	-40
$20\log\frac{1}{j\omega}$ dB	40	20	0	-20	-40

$20\log\frac{1}{j\omega+1}$ dB	0	0	-3	-20	-40
Bode magnitude, dB	80	40	-3	-60	-120

Example 12.2

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s) = \frac{1}{s^4 + 5s^3 + 8s^2 + 6s}$$

Solution

$$G(s)H(s) = \frac{1}{s(s^2 + 2s + 2)(s + 3)}$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega\left((j\omega)^2 + 2(j\omega) + 2\right)((j\omega) + 3)}$$

$$G(j\omega)H(j\omega) = \frac{\frac{1}{3}}{j\omega\left((2-\omega^2) + j2\omega) + 2\right)(j\frac{\omega}{3} + 1)}$$

Comparing the second order term with a standard second order term,

$$\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n$$

$\omega_n = \sqrt{2}$ and $\zeta = \frac{1}{\sqrt{2}}$.

For the first order integral factor, ω_c=3 rad/s

For ζ> 0.5, the response at resonance is less than theresponse at frequencies less than the resonant frequencies

Table Computation of Bode magnitude using asymptotic properties of the integral second-order term

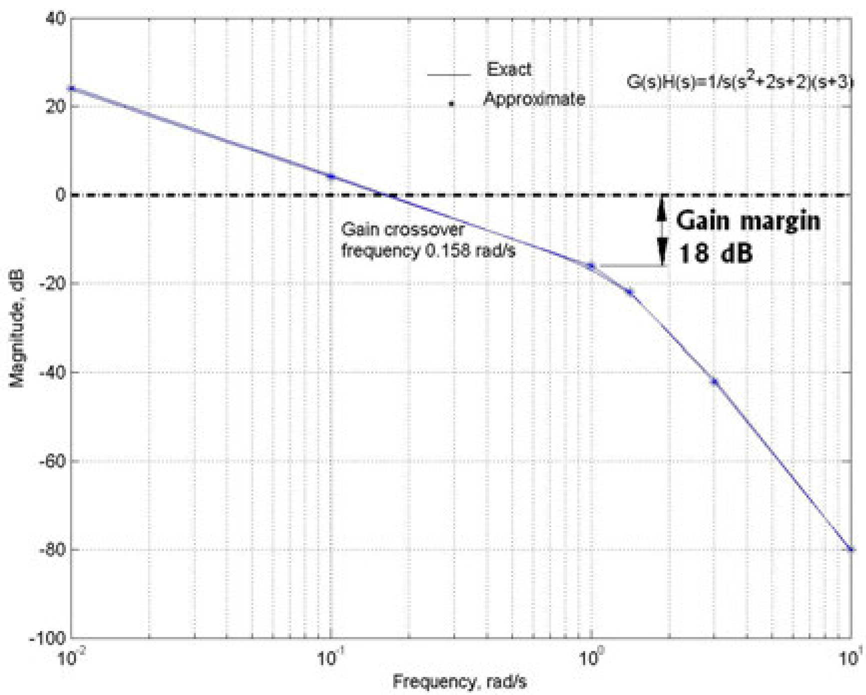
	x1	x10		x1	x2		x3	x1		x1	x10		x3	x1
Frequency, rad/s	1.4	14		14	30		30	10		10	100		30	3
Magnitude, dB	-6	-46		-46	-58		-58	-38		-38	-78		-58	-18

Table Computation of Bode magnitude using asymptotic properties of the integral first-order term

	x1	x3		x2	x1		x3	x1		x1	x10
Frequency, rad/s	3	30		30	14		30	10		10	100
Magnitude, dB	0	-20		-20	-14		-20	-10		-10	-30

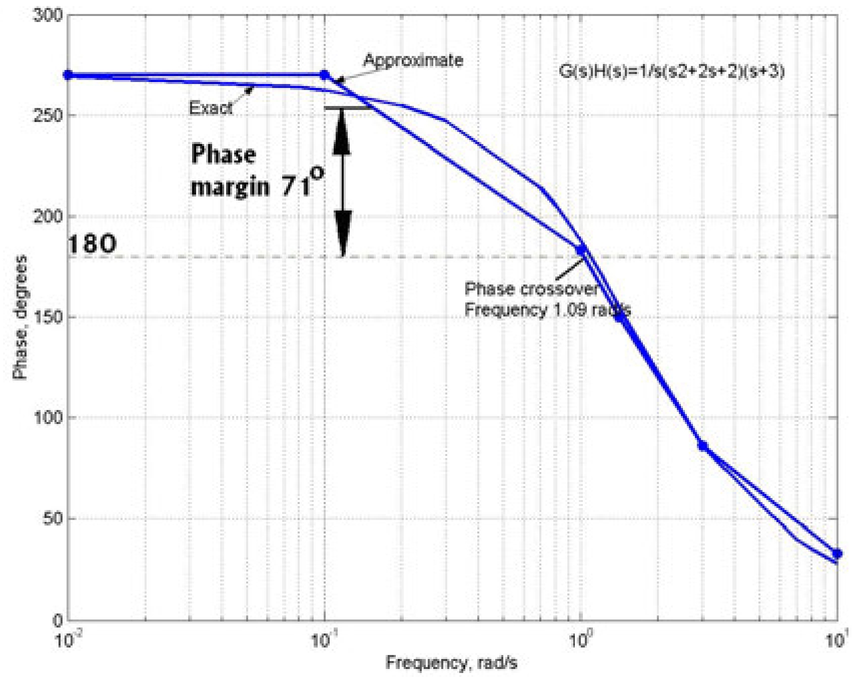
Bode magnitude

	Frequency, rad/s										
						ω_n	ω_c				
	0.01	0.1	0.14	0.3	1	$\sqrt{2}$	3	10	14	30	100
$20\log\frac{1}{3}$	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
$20\log\frac{1}{j\omega}$	40	20	17	10	0	-3	-10	-20	-23	-30	-40
$20\log\frac{1}{\left((2-\omega^2)+j(2\omega)\right)}$	-6	-6	-6	-6	-6	-9	-18	-38	-46	-58	-78
$20\log\frac{1}{j\frac{\omega}{3}+1}$	0	0	0	0	0	-1	-3	-10	-14	-20	-30
Bode magnitude, dB	24	4	1	-6	-16	-23	-41	-78	-93	-118	-158

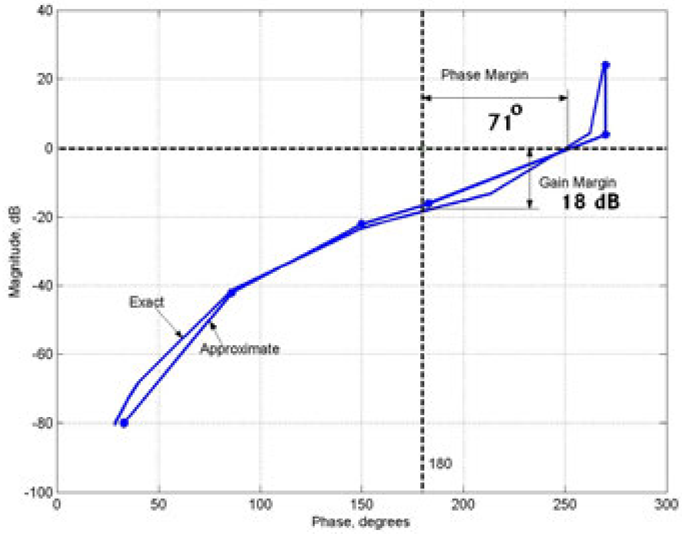


Bode phase

	Frequency, rad/s										
						ω_n	ω_c				
	0.01	0.1	0.14	0.3	1	$\sqrt{2}$	3	10	14	30	100
$\angle\frac{1}{3}$	0	0	0	0	0	0	0	0	0	0	0
$\angle\frac{1}{j\omega}$ degrees	270	270	270	270	270	270	270	270	270	270	270
$\angle\frac{1}{\left((2-\omega^2)+j(2\omega)\right)}$ degrees	360	360	360	343	297	270	221	192	180	180	180
$\angle\frac{1}{j\frac{\omega}{3}+1}$, degrees	360	360	360	360	336	330	315	291	285	270	270
Bode phase, degrees	270	270	250	253	183	150	86	33	15	0	0



Nichols plot



13. Polar Plots

It is a graphical method of determining stability of feedback control systems by using the polar plot of their open-loop transfer functions.

Example 13.1

Draw a polar plot of the open-loop transfer functionfor

$$G(s)H(s) = \frac{K}{s(s+1)} \tag{14.33}$$

Frequency response

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)} \tag{14.34}$$

Magnitude

$$\left|G(j\omega)H(\omega)\right| = \frac{K}{\omega\sqrt{1+\omega^2}} \tag{14.35}$$

Angle

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\omega \tag{14.36}$$

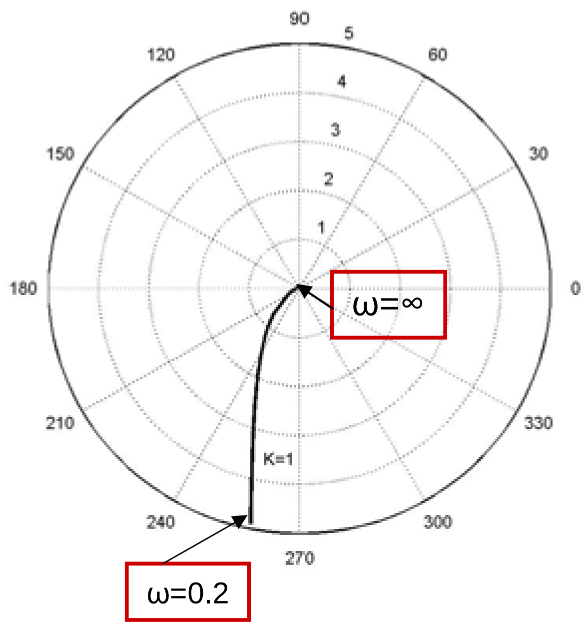
$$270^0 < \angle G(j\omega)H(j\omega) < 180^0 \tag{14.37}$$

Magnitude and phaseof the open-loopfrequency transfer function

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0	∞	270

2	**0.2**	**4.9029**	**259**
3	**0.4**	**2.3212**	**248**
4	**0.8**	**0.9761**	**231**
5	**1**	**0.7071**	**225**
6	**4**	**0.0606**	**194**
7	**10**	**0.01**	**186**
8	**50**	**0.0004**	**181**
9	**100**	**0.0001**	**181**
10	**200**	**≈0**	**≈180**

Polar plot of the transfer function $\frac{K}{s(s+1)}$ and K=1



Example 14.2
Draw a polar plot of the open-loop transfer functionfor K=1, 10, 25, 55

$$GH = \frac{K}{s(s+2)(s+4)}$$

Solution
Frequency response

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+4)}$$

Magnitude

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{\omega^2+4}\sqrt{\omega^2+16}}$$

Angle

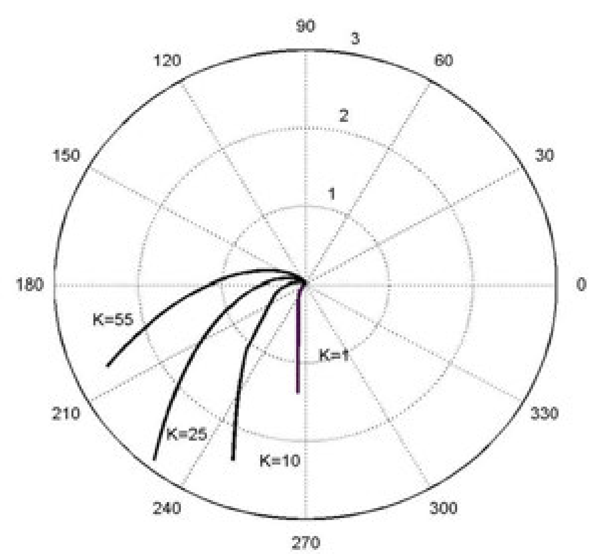
$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

The lies in II and III quadrants as $90^0 < \angle G(j\omega)H(j\omega) < 270^0$

Magnitude and phase of the open-loop frequencytransfer function (K=1)

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0.1	1.2481	266
2	0.2	0.6211	261
4	0.4	0.3049	253
5	0.8	0.1423	237
6	1	0.1085	229
7	4	0.0099	162
8	10	0.0009	123
9	50	0	97

Polar plot of the transfer function $GH = \frac{K}{s(s+2)(s+4)}$ for K=1, 10, 25, 55



Example 14.3

Draw a polar plot of the open-loop transfer function $G(s)H(s) = \frac{K}{s^2(s+1)}$

Solution

Frequency response

$$G(j\omega)H(j\omega)=\frac{K}{(j\omega)^2(j\omega+1)}$$

Magnitude

$$\left|G(j\omega)H(j\omega)\right|=\frac{K}{\omega^2\sqrt{\omega^2+1}}$$

Angle

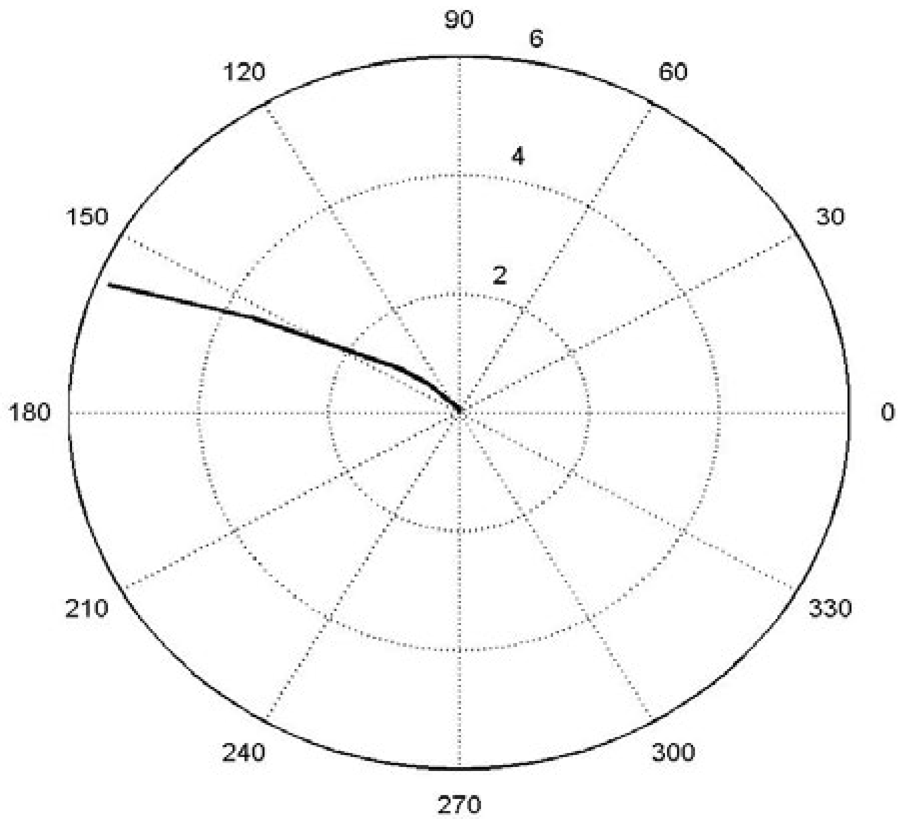
$$\angle G(j\omega)H(j\omega)=-180^0-\tan^{-1}\omega$$

The lies in II quadrant only as $90^0<\angle G(j\omega)H(j\omega)<180^0$

Magnitude and phase of the open-loop frequency transfer function (K=1)

No.	**Frequency, rad/s**	**Magnitude**	**Phase, degrees**
1	0.4	5.803	158
2	0.5	3.5777	153
4	0.8	1.2201	141
5	1	0.7071	135
6	2	0.1118	117
7	3	0.0351	108
8	4	0.0152	104
9	5	0.0078	101

Polar plot of the transfer function $GH=\frac{K}{s(s+2)(s+4)}$ for K=1, 10, 25, 55



Equation Chapter (Next) Section 1

Bode plot using MATLAB

Program 1: Sketch the bode plot for the open loop transfer function $\frac{C(s)}{R(s)} = \frac{40}{s(s+1)(s+4)}$.

Determine the gain margin, phase margin, gain cross over frequency and phase cross over frequency.

Solution:

```
>> num=[0 40]
num=
0        40
>> q1=[1 0];
>> q2=[1 1];
>> q3=[1 4];
>> den=conv(q1,q2);
>> den=conv(den,q3);
den=
1        9        24        16        0
>>sys=tf(num,den)
Transfer function:
        1
--------------------
   s^4+9s^3+24s^2+16s
>>bode(sys)
>>margin(sys)
```

14. Nyquist plot

14.1. Definition

Nyquist criterion is a graphical method of determining stability of feedback control systems by using the Nyquist plot of their open-loop transfer functions.

14.2. Theory

Feedback transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{14.1}$$

Poles and zeros of theopen-loop transfer function

$$G(s)H(s) = \frac{K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{14.2}$$

$$1+G(s)H(s) = \frac{(s-p_1)(s-p_2)...(s-p_n)+K(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{14.3}$$

Number of closed-loop poles - Number of zeros of 1+GH = N umber of open-loop poles

$$1+G(s)H(s) = \frac{(s-z_{c_1})(s-z_{c_2})(s-z_{c_n})}{(s-p_1)(s-p_2)...(s-p_n)} \tag{14.4}$$

$z_{c_1}, z_{c_2}...z_{c_n}$ = zeros of 1+G(s)H(s)

These are also poles of the close-loop transfer function

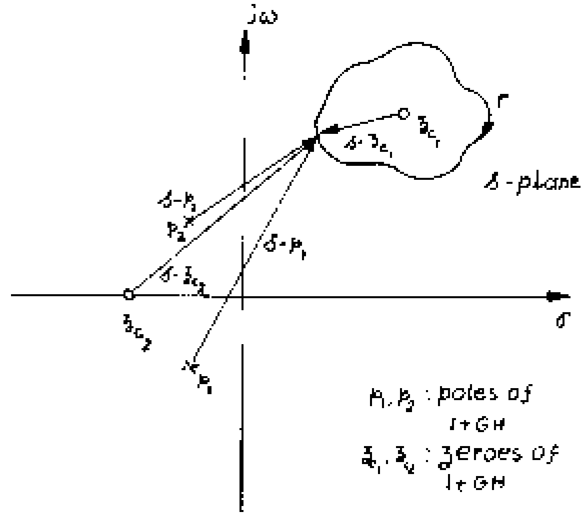
Magnitude

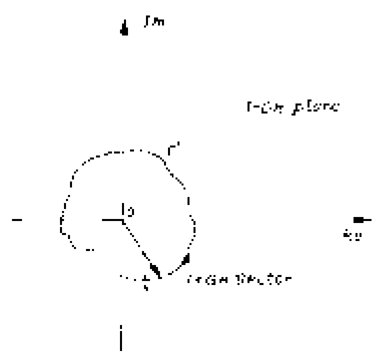
$$\left|1+G(s)H(s)\right| = \frac{\left|s-z_{c_1}\right|\left|s-z_{c_2}\right|...\left|s-z_{c_n}\right|}{\left|(s-p_1)\right|\left|(s-p_2)\right|...\left|(s-p_n)\right|.} \tag{14.5}$$

Angle

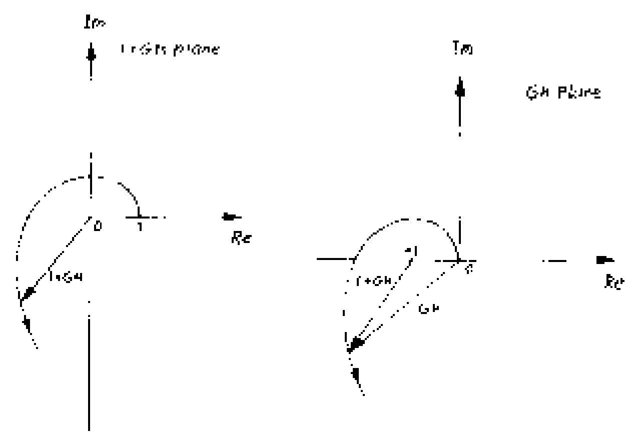
$$\angle 1+G(s)H(s) = \frac{\angle s-z_{c_1}\angle s-z_{c_2}\angle s-z_{c_n}}{\angle(s-p_1)\angle(s-p_2)\angle(s-p_n)} \tag{14.6}$$

The s-plane to 1+GH plane mappingphase angle of the 1+G(s)H(s) vector, corresponding to a point on the s-plane is the difference between the sum of the phase of all vectors drawn from zeros of 1+GH(close loop poles) and open loops on the s plane. If this point s is moved along a closed contour enclosing any or all of theabove zeros and poles, only the phase of the vector of each of the enclosed zeros or open-loop poles will change by 360^0. The directionwill be in the same sense of the contour enclosing zeros and in the opposite sense for the contour enclosing open-loop poles.

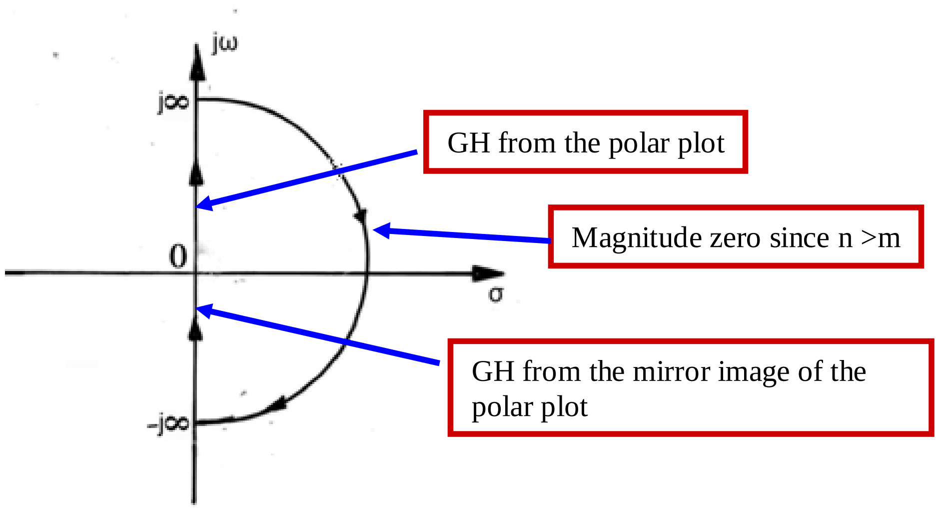




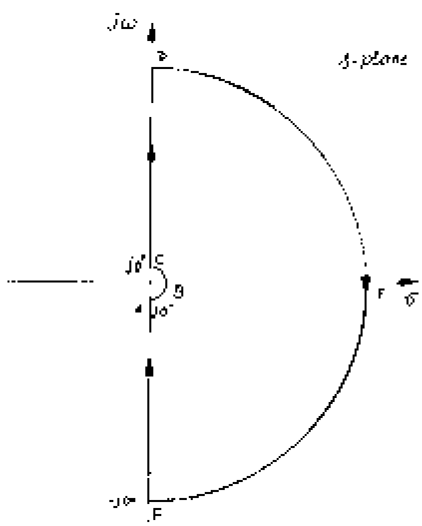
14.3.Principle of argument



When a closed contour in the s-plane encloses a certain number of poles and zeros of 1+G(s)H(s) in the clockwise direction, the number ofencirclements of the origin by the corresponding contour in the G(s)H(s)plane will encircle the point (-1,0) a number of times given by thedifference between the number of its zeros and poles of 1+G(s)H(s) it enclosed on the s-plane.



Modified contour on the s-plane forchecking the existence of closed-looppoles
$s = \varepsilon e^{j\beta}$



Magnitude of GH remains the same alongthe contourPhase of β changes from 270 to 90 degrees

14.4.Gain Margin and Phase Margin

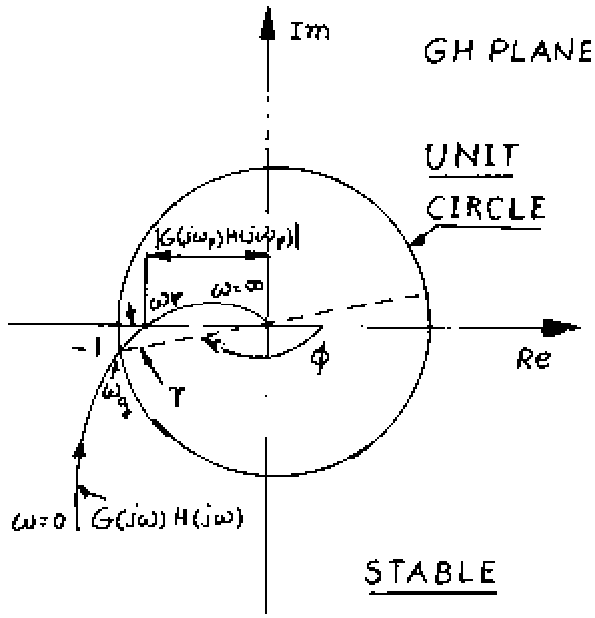
Phase crossover frequency ω_p is the frequency at which the open-loop transfer function has a phase of 180^0. The gain crossover frequency ω_g is the frequency at whichthe open-loop transfer function has a unit gain

Gain margin

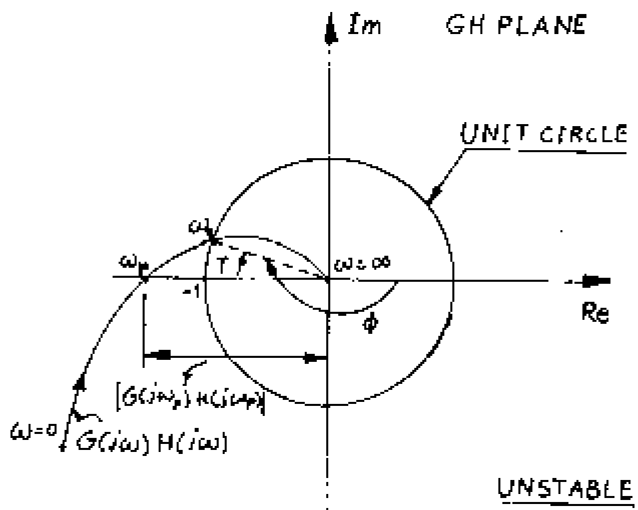
$$M = -20\log\left|G(j\omega_p)H(j\omega_p)\right| \tag{14.7}$$

Phase margin

$$\gamma = \angle G(j\omega_g)H(j\omega_g) - 180^o \tag{14.8}$$



(b)



14.5.Procedure

(1) Locate open-loop poles on the s-plane
(2) Draw the closed contour and avoid open-loop poles on the imaginary axis
(3) Count the number of open-loop poles enclosed in the above contour of step 2, say P
(4) Plot G(jω)H(jω) and its reflection on the GH plane and map part of the small semi-circle detour on the s-plane around poles (if any) on the imaginary axis.
(5) Once the entire s-plane contour is mapped on to the GH plane, count the number of encirclements of the point (-1,0) and its direction. Clockwise encirclement is considered positive, say N.
(6) The number of closed-loop poles in the right-half s-plane is given by Z=N+P. if Z >0, the system is unstable.
(7) Determine gain margin, phase margin, and critical value of open-loop gain.

Example 14.1

Using Nyquist criterion, determine the stability of a feedback systemwhose open-loop transfer function is given by

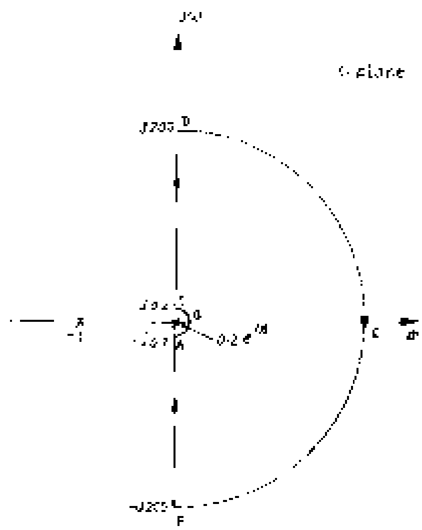
$$G(s)H(s) = \frac{K}{s(s+1)}$$

Solution

Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0 and –1. Let K=1

Step 2 Draw the closed contour on the s-plane to check the existenceof closed-loop poles in the right-half s-plane.

Open-loop poles and s-plane contour

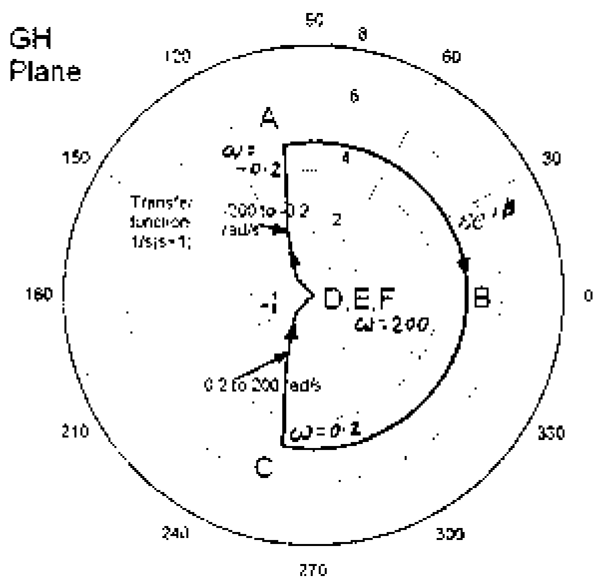


$$|G(j\omega)H(\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\omega$$

No.	Frequency, rad/s		Magnitude	Phase, degrees	β, s-plane, deg	β, GH plane, deg
1	0.2	Positive frequencies	4.9029	259	270	101
2	0.4		2.3212	248	280	91
3	0.8		0.9761	231	290	80
4	1		0.7071	225	300	69
5	4		0.0606	194	310	58
6	10		0.01	186	320	46
7	50		0.0004	181	330	35
8	100		0.0001	181	340	23

9	200		0	180	350	12
10	-200	Negative frequencies	0	180	0	0
11	-100		0.0001	179	10	348
12	-50		0.0004	179	20	337
13	-10		0.01	174	30	325
14	-4		0.0606	166	40	314
15	-1		0.7071	135	50	302
16	-0.8		0.9761	129	60	291
17	-0.4		2.3212	112	70	280
18	-0.2		4.9029	101	80	269

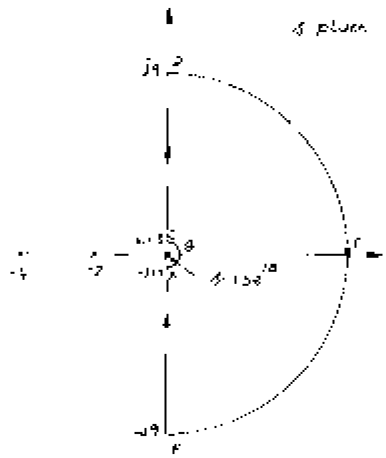


The above system is stable. Here, phase crossover frequency is very large (infinity) and gain crossover frequency 0.786 rad/s. Phase angle corresponding to gain crossover frequency= $232^0$and Phase margin is 52^o

Example 14.2.
Using Nyquist criterion, determine the stability of a feedback systemwhose open-loop transfer function is given by

$$G(s)H(s) = \frac{55}{s(s+2)(s+4)}$$

Solution
Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0, -2 and –4. Let K=1
Step 2 Draw the closed contour on the s-plane to check the existenceof closed-loop poles in the right-half s-plane.
Open-loop poles and s-plane contour



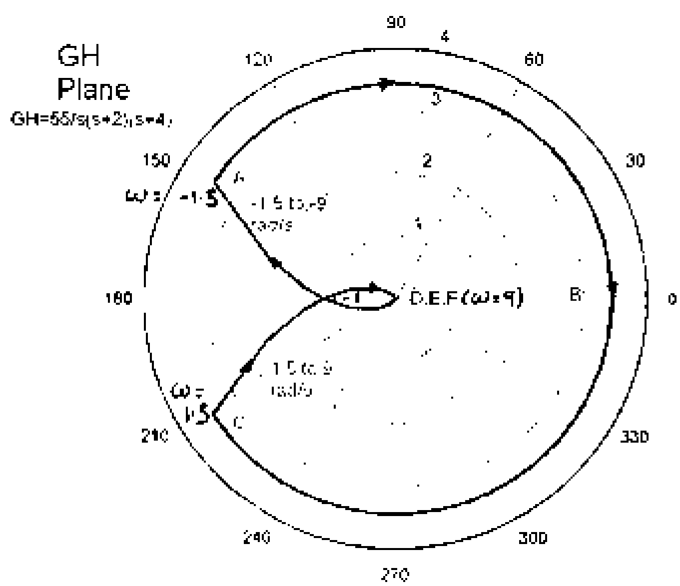
The number of open-loop pole enclosed, P is zero

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{\omega^2+4}\sqrt{\omega^2+16}}$$

$$\angle G(j\omega)H(j\omega) = -\frac{\pi}{2} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{4}$$

No.	Frequency		Magnitude	Phase, degrees	β, s-plane, deg
1	1.5	Positive frequencies	3.4332	213	270
2	2		2.1741	198	280
3	2.5		1.4568	187	290
4	2.83		1.1446	180	300
5	3		1.017	177	310
6	3.5		0.7334	169	320
7	4.5		0.4122	156	330
8	5		0.319	150	340
9	5.5		0.2513	146	350
10	6		0.201	142	0
11	7		0.1339	136	10
12	8		0.0932	131	20
13	9		0.0673	126	30
14	-9	Negative frequencies	0.0673	234	40
15	-8		0.0932	229	50
16	-7		0.1339	224	60

17	-6		0.201	218	70
18	-5.5		0.2513	214	80
19	-5		0.319	210	90
20	-4.5		0.4122	204	0
21	-3.5		0.7334	191	343
22	-3		1.017	183	326
23	-2.83		1.1446	180	309
24	-2.5		1.4568	173	292
25	-2		2.1741	162	276
26	-1.5		3.4332	147	259



Here, Z=N+P=2.
Hence, the above system is unstable.
Again,
Phase crossover frequency 2.83 rad/s
The gain at which the system becomes marginally stable, $K^* = 55/1.1446 = 48$
Gain margin

$$M = -20\log\left|G(j\omega_p)H(j\omega_p)\right|$$
$$= -20\log\left|1.1446\right| = -1.17\,\text{dB}$$

Gain crossover frequency =3 rad/s and the corresponding angle of GH=177^o
Phase margin=177-180=-3^o

Nyquist plot using MATLAB

Program 1: Sketch the nyquist plot for the open loop transfer function $\frac{C(s)}{R(s)} = \frac{40}{s(s+1)(s+4)}$.

Solution:

```
>> num=[040]
num=
0          40
>> q1=[1 0];
>> q2=[1 1];
>> q3=[1 4];
>> den=conv(q1,q2);
>> den=conv(den,q3);
den=
1          9          24         16         0
>>sys=tf(num,den)
Transfer function:
        1
--------------------
   s^4+9s^3+24s^2+16s
>>nyquist(sys)
```