



KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -II

PART-3

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CHAPTER-5

LIMIT

Some definitions Related to limit

Constant: A quantity which does not change its value under different mathematical operations of a problem is called constant.

Ex: $5, \pi, e, \sqrt{3}$ etc. are all constants.

Variable: A quantity which changes its value under different mathematical operations of a problem is called a variable.

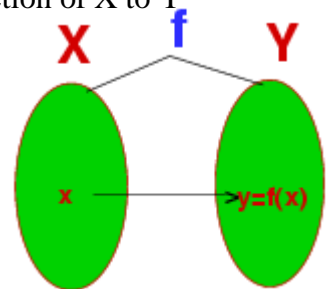
Ex: The equation of a straight line $y = mx + c$. Here x & y are variables.

Independent variable: A variable (x), which can assume any arbitrary value from a set is called an independent variable.

Dependent variable: A variable (y), whose value depends on the value of another independent variable (x) is called dependent variable.

Ex: Area of a circle is $A = \pi r^2$. Here A is the dependent variable & r is an independent variable.

Function or Mapping: Let X and Y be any two non empty sets and there be a correspondence between the elements of X and Y such that for every element $x \in X$, there exists a unique and definite element $y \in Y$, which is written as $y = f(x)$, then we say that f is a function of X to Y , written as $f : X \rightarrow Y$.



Note-1: Every element of X will be mapped by f .

2: The element of $y \in Y$ is called the image of x under f

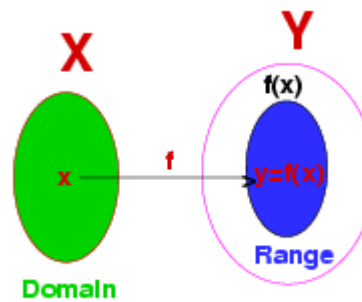
Meaning of $f(a)$: The value of the function $y = f(x)$ at $x = a$ denoted by $f(a)$. It is obtained by putting a in place of x in $f(x)$.

Domain and range of a function: Let $f : X \rightarrow Y$ be a mapping from a set X to the set Y .

Domain: The domain of $f = X$

Range: The range of $f = f(x) = \{f(x) : f(x) \in Y, x \in X\}$ = set of all image points in Y .

- Note: 1. If $x \in X$, then $f(x) \in f(X)$
 2. $f(X) \subseteq Y$ i.e. Range of $f \subseteq Y$
 3. Y is called the co-domain of f

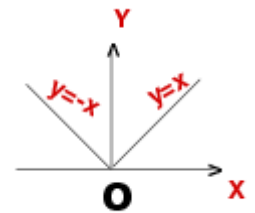


Real valued function: If $f : X \rightarrow Y$ be a function from a set X to a set Y where $X, Y \subseteq \mathbb{R}$, then f is a real valued function.

Ex: $y = f(x) = \frac{x+1}{x^2+1}, x \in \mathbb{R}$

Modulus function: The modulus of a real number x denoted by $|x|$ is its magnitude or numerical value taken with a positive sign.

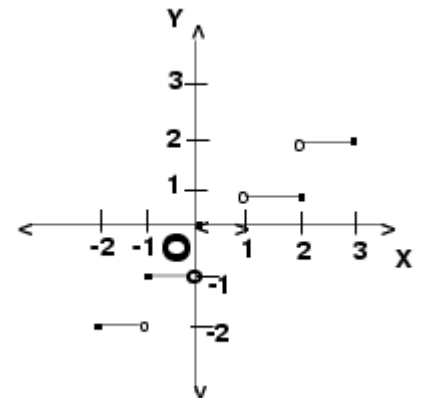
$$\text{i.e. } |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Greatest integer function: The greatest integer function of x is denoted by $[x]$ is the greatest integer less than or equal to x i.e. $[x] \leq x$.

$$y = [x] = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \dots \dots \dots \text{and so on.} \end{cases}$$

Ex: $[3] = 3, [-3] = -3, [0] = 0,$



Ex: $[2] = 2, [-3] = -3, [0] = 0, [2.98] = 2, [-2.98] = -3, [0.5] = 0, [-0.5] = -1,$

$[2-h] = 1, [2+h] = 2$ where h is a small +ve quantity.

Logarithm function: If $a > 0$ and $a \neq 1$ then $y = \log_a x$ is called a logarithm function with base a , for all $x > 0$.

Note: 1. $\log_a a = 1$, $\log_a 1 = 0$, $\log_e x = \ln x$

2. $\log_a 0 = \begin{cases} -\infty, & \text{if } a > 0 \\ +\infty, & \text{if } 0 < a < 1 \end{cases}$

Ex: $y = \log_e(x+1)$, $y = \log_3(x^2+2)$

Exponential function: The function $y = a^x$ or $y = e^x$ is called an exponential function, for all $x \in \mathbb{R}$.

Ex: $y = 2^x$, $y = x^{\sin x}$, $y = e^{\cos x}$

Note: 1: $e^{\ln x} = x = e^{\log_e x}$, $a^x = e^{x \log_e a}$, $a^{\log_a x} = x$, ($a > 0, a \neq 1$)

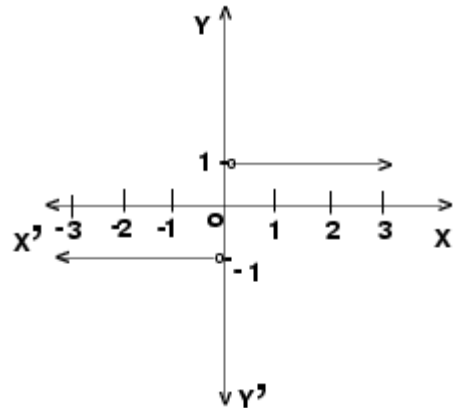
2: $\log_e mn = \log_e m + \log_e n$

3: $\log_e \frac{m}{n} = \log_e m - \log_e n$

4: $\log_e m^n = n \log_e m$

Signum function: The signum function is defined as

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



LIMIT OF A FUNCTION The limit of a function $f(x)$

is defined as the value of a function at some indicated points.

The limit or limiting value of a function is the fundamental concept of calculus.

Definition: If 'l' be the limiting value of a function $f(x)$ at a point $x = a$ then which is written as

$$\lim_{x \rightarrow a} f(x) = l \text{ OR } \lim_{x \rightarrow a} \left(\begin{array}{l} \text{which is read as the value of } f(x) \text{ tends to } l \text{ as } x \text{ tends to } a \\ \text{i.e } f(x) \rightarrow l \text{ as } x \rightarrow a \end{array} \right)$$

Note: $x \rightarrow a$ means x close to a but $x \neq a$

Limit or limiting value of a function: (Meaning of $\lim_{x \rightarrow a}$)

If $y = f(x)$ be a given function of x then for every value of x we obtain a definite and unique value of. But sometimes for a particular value of x , the value of $f(x)$ is not determinate.

For example Let $y = f(x) = \frac{x^2 - 9}{x - 3}$ at $x = 3$

$$f(3) = \frac{3^2 - 9}{3 - 3} = \frac{0}{0} \text{ which is not determinate.}$$

In maximum case we obtain the limiting value of the function at a point where it is indeterminate.

Algebra of limits

1. $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

2. $\lim_{x \rightarrow a} (f(x) \times g(x)) = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$

3. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ where $\lim_{x \rightarrow a} g(x) \neq 0$

4. $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x)$ Where k is a constant.

5. $\lim_{x \rightarrow a} k = k$ Where k is a constant

LIMIT FORMULAS

F - 1. For n be any rational

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$F - 2. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$F - 3. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \text{ or } \log_e a \ (a > 0)$$

$$F - 4. \lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{x}{\ln(1+x)} = 1$$

$$F - 5. \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$F - 6. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$F - 7. \lim_{x \rightarrow 0} (1 + \lambda x)^{\frac{1}{x}} = e^\lambda$$

$$F - 8. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$F - 9. \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{x}{\sin^{-1} x} = 1$$

$$F - 10. \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1$$

$$F - 11. \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \text{ or } \lim_{x \rightarrow 0} \frac{x}{\tan^{-1} x} = 1$$

$$F - 12. \lim_{x \rightarrow 0} \sin x = 0, \lim_{x \rightarrow 0} \cos x = 1$$

$$F - 13. \lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n$$

$$F - 14. \lim_{x \rightarrow 0} \frac{(1-x)^n - 1}{x} = -n$$

Rules for finding $\lim_{x \rightarrow a} f(x)$

Rule -1. Put $x = a$ If $f(a)$ gives a finite value

$$\text{Ex: } \lim_{x \rightarrow 2} (x^2 + 3x + 1)$$

$$\text{Ans: } \lim_{x \rightarrow 2} (x^2 + 3x + 1) = 2^2 + 3 \times 2 + 1 = 11$$

$$\text{Ex: } \lim_{x \rightarrow 3} (x^2 + 3x)(x - 2)$$

$$\text{Ans: } \lim_{x \rightarrow 3} (x^2 + 3x)(x - 2) = \lim_{x \rightarrow 3} (x^2 + 3x) \times \lim_{x \rightarrow 3} (x - 2) = (3^2 + 3 \times 2) \times (3 - 2) = 15 \times 1 = 15$$

$$\text{Ex: } \lim_{x \rightarrow 1} \frac{x^2 - 2x + 5}{x + 3} = \frac{\lim_{x \rightarrow 1} (x^2 - 2x + 5)}{\lim_{x \rightarrow 1} (x + 3)} = \frac{1^2 - 2 \times 1 + 5}{1 + 3} = \frac{4}{4} = 1$$

Rule -2. $\left(\frac{0}{0}\right)$ form

If $f(x)$ be a rational function then factorize both numerator (N') & denominator (D'). cancel the common factor and put $x = a$.

$$\text{Ex: } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$$

$$\text{Ans: } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$$

$$\text{Ex: } \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$$

$$\text{Ans: } \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 1) = 1 + 1 + 1 = 3$$

$$\text{Ex: } \lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x - 2}$$

$$\begin{aligned} \text{Ans: } \lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x - 2} &= \lim_{x \rightarrow 2} \frac{x^2 - 3x - 2x + 6}{x - 2} = \lim_{x \rightarrow 2} \frac{x(x - 3) - 2(x - 3)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x - 3)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x - 3) = 2 - 3 = -1 \end{aligned}$$

Rule-3. If the given function contains a surd, then multiply a conjugate surd in both N^r & D^r, after simplification put $x = a$.

$$Ex: \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$$

$$\begin{aligned} Ans: \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{(x+2-2)}{x(\sqrt{x+2} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{x+2} + \sqrt{2})} = \frac{1}{\sqrt{2} + \sqrt{2}} \text{ (put } x = 0) \\ &= \frac{1}{2\sqrt{2}} \end{aligned}$$

$$Ex: \lim_{x \rightarrow 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$$

$$\begin{aligned} Ans: \lim_{x \rightarrow 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{2-x} - \sqrt{2+x})(\sqrt{2-x} + \sqrt{2+x})}{x(\sqrt{2-x} + \sqrt{2+x})} \\ &= \lim_{x \rightarrow 0} \frac{(2-x) - (2+x)}{x(\sqrt{2-x} + \sqrt{2+x})} = \lim_{x \rightarrow 0} \frac{2-x-2-x}{x(\sqrt{2-x} + \sqrt{2+x})} = \lim_{x \rightarrow 0} \frac{-2x}{x(\sqrt{2-x} + \sqrt{2+x})} = \frac{-2}{(\sqrt{2-x} + \sqrt{2+x})} \\ &= \frac{-2}{\sqrt{2} + \sqrt{2}} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} \end{aligned}$$

$$Ex - \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} \cdot Ex - \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+4} - 2} \cdot Ex - \lim_{x \rightarrow 0} \frac{\sqrt{x^2+4} - 2}{x^2} \cdot Ex - \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x}$$

$$Ex - \lim_{x \rightarrow a} \frac{\sqrt{x-b} - \sqrt{a-b}}{x^2 - a^2}$$

$$Ex: \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$$

$$\begin{aligned} Ans: \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \frac{(x^2 - \sqrt{x}) \times (x^2 + \sqrt{x}) \times (\sqrt{x} + 1)}{(\sqrt{x} - 1) \times (\sqrt{x} + 1) \times (x^2 + \sqrt{x})} \text{ (double conjugate multiplication)} \\ &= \lim_{x \rightarrow 1} \frac{(x^4 - x)(\sqrt{x} + 1)}{(x-1)(x^2 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{x(x^3 - 1)(\sqrt{x} + 1)}{(x-1)(x^2 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{x(x-1)(x^2 + x + 1)(\sqrt{x} + 1)}{(x-1)(x^2 + \sqrt{x})} \\ &= \lim_{x \rightarrow 1} \frac{x(x^2 + x + 1)(\sqrt{x} + 1)}{(x^2 + \sqrt{x})} = \frac{1 \times 3 \times 2}{2} = 3 \end{aligned}$$

$$Ex - \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{\sqrt{x+3} - \sqrt{3}} \cdot Ex - \lim_{x \rightarrow 0} \frac{\sqrt{x^2+1} - 1}{\sqrt{x^2+4} - 2}$$

$$\text{Formula : } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\text{Ex : } \lim_{x \rightarrow 2} \frac{x^9 - 2^9}{x - 2}$$

$$\text{Ans : } \lim_{x \rightarrow 2} \frac{x^9 - 2^9}{x - 2} = 9 \times 2^8$$

$$\text{Ex : } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$$

$$\text{Ans : } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} = \lim_{x \rightarrow 2} \frac{x^5 - 2^5}{x - 2} = 5 \times 2^4 = 32$$

$$\text{Ex : } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8}$$

$$\text{Ans : } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8} = \frac{\lim_{x \rightarrow 2} \frac{x^5 - 2^5}{x - 2}}{\lim_{x \rightarrow 2} \frac{x^3 - 2^3}{x - 2}} = \frac{5 \times 2^4}{3 \times 2^2} = \frac{32}{12} = \frac{8}{3}$$

$$\text{Ex : } \lim_{x \rightarrow 1} \frac{x^n - 1}{x^m - 1}$$

$$\text{Ans : } \lim_{x \rightarrow 1} \frac{x^n - 1}{x^m - 1} = \frac{\lim_{x \rightarrow 1} \frac{x^n - 1^n}{x - 1}}{\lim_{x \rightarrow 1} \frac{x^m - 1^m}{x - 1}} = \frac{n \times 1^{n-1}}{m \times 1^{m-1}} = \frac{n}{m}$$

$$\text{Ex : } \lim_{x \rightarrow 5} \frac{\sqrt{x} - \sqrt{5}}{x - 5}, \text{ Ex : } \lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}, \text{ Ex : } \lim_{x \rightarrow 3} \frac{x^{\frac{1}{2}} - 1}{x^m - 1}, \text{ Ex : } \lim_{x \rightarrow b} \frac{x^8 - b^8}{x^3 - b^3}$$

$$\text{Formula : } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{ax} \times a = 1 \times a = a$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{5x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{5x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{-3x} \times -3 = \frac{1}{5} \times 1 \times (-3) = \frac{-3}{5}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{e^{5x} - 1}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{e^{5x} - 1} = \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{x}}{\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x}} = \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{-3x} \times -3}{\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{5x} \times 5} = \frac{1 \times (-3)}{1(5)} = \frac{-3}{5}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{ax} - e^{-ax}}{x}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{e^{ax} - e^{-ax}}{x} &= \lim_{x \rightarrow 0} \frac{e^{ax} - 1 + 1 - e^{-ax}}{x} = \lim_{x \rightarrow 0} \frac{(e^{ax} - 1) - (e^{-ax} - 1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} - \lim_{x \rightarrow 0} \frac{e^{-ax} - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{ax} \times a - \lim_{x \rightarrow 0} \frac{e^{-ax} - 1}{-ax} \times (-a) = 1 \times a - 1 \times (-a) = 2a \end{aligned}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{-3x} - e^{4x}}{e^{5x} - e^{-2x}}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{e^{-3x} - e^{4x}}{e^{5x} - e^{-2x}} &= \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - e^{4x}}{x}}{\lim_{x \rightarrow 0} \frac{e^{5x} - e^{-2x}}{x}} = \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - 1 + 1 - e^{4x}}{x}}{\lim_{x \rightarrow 0} \frac{e^{5x} - 1 + 1 - e^{-2x}}{x}} = \frac{\lim_{x \rightarrow 0} \frac{(e^{-3x} - 1) - (e^{4x} - 1)}{x}}{\lim_{x \rightarrow 0} \frac{(e^{5x} - 1) - (e^{-2x} - 1)}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{x} - \lim_{x \rightarrow 0} \frac{(e^{4x} - 1)}{x}}{\lim_{x \rightarrow 0} \frac{(e^{5x} - 1)}{x} - \lim_{x \rightarrow 0} \frac{(e^{-2x} - 1)}{x}} = \frac{\lim_{x \rightarrow 0} \frac{e^{-3x} - 1}{-3x} \times (-3) - \lim_{x \rightarrow 0} \frac{(e^{4x} - 1)}{4x} \times 4}{\lim_{x \rightarrow 0} \frac{(e^{5x} - 1)}{5x} \times 5 - \lim_{x \rightarrow 0} \frac{(e^{-2x} - 1)}{-2x} \times (-2)} = \frac{-3 - 4}{5 - (-2)} = \frac{-7}{7} = -1 \end{aligned}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}, \text{Ex : } \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{e^{-3x} - 1}, \text{Ex : } \lim_{x \rightarrow 0} \frac{e^{3x} - e^{5x}}{x}, \text{Ex : } \lim_{x \rightarrow 0} \frac{e^{ax} - e^{cx}}{e^{bx} - e^{dx}}$$

$$\text{Formula : } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \text{ or } \log_e a (a > 0)$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{3^x - 1}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{3^x - 1}{x} = \ln 3 \text{ or } \log_e 3$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{a^{2x} - 1}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{a^{2x} - 1}{x} = \lim_{x \rightarrow 0} \frac{a^{2x} - 1}{2x} \times 2 = \ln a \times 2 = 2 \ln a = \ln a^2$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{b^{3x} - 1}{5x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{b^{3x} - 1}{5x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{b^{3x} - 1}{x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{b^{3x} - 1}{3x} \times 3 = \frac{1}{5} \times \ln b \times 3 = \frac{3}{5} \times \ln b$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1} = \frac{\lim_{x \rightarrow 0} \frac{a^x - 1}{x}}{\lim_{x \rightarrow 0} \frac{b^x - 1}{x}} = \frac{\ln a}{\ln b} = \log_b a$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{a^x - b^x}{x} &= \lim_{x \rightarrow 0} \frac{a^x - 1 + 1 - b^x}{x} = \lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \\ &= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} - \lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \ln a - \ln b = \ln \frac{a}{b} \end{aligned}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{a^x - b^x}{c^x - d^x}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{a^x - b^x}{c^x - d^x} &= \frac{\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}}{\lim_{x \rightarrow 0} \frac{c^x - d^x}{x}} = \frac{\lim_{x \rightarrow 0} \frac{a^x - 1 + 1 - b^x}{x}}{\lim_{x \rightarrow 0} \frac{c^x - 1 + 1 - d^x}{x}} = \frac{\lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x}}{\lim_{x \rightarrow 0} \frac{(c^x - 1) - (d^x - 1)}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \frac{a^x - 1}{x} - \lim_{x \rightarrow 0} \frac{b^x - 1}{x}}{\lim_{x \rightarrow 0} \frac{c^x - 1}{x} - \lim_{x \rightarrow 0} \frac{d^x - 1}{x}} = \frac{\ln a - \ln b}{\ln c - \ln d} = \frac{\ln \frac{a}{b}}{\ln \frac{c}{d}} \end{aligned}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{3^x - 1}{5^x - 1}, \text{Ex : } \lim_{x \rightarrow 0} \frac{3^x - 5^x}{x}, \text{Ex : } \lim_{x \rightarrow 0} \frac{3^x - 4^x}{5^x - 7^x}$$

$$\text{Formula : } \lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{x} = \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{2x} \times 2 = 1 \times 2 = 2$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{5x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{5x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{\log_e(1+2x)}{2x} \times 2 = \frac{1}{5} \times 1 \times 2 = \frac{2}{5}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\log_e(1-3x)}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\log_e(1-3x)}{x} = \lim_{x \rightarrow 0} \frac{\log_e\{1+(-3)x\}}{(-3)x} \times (-3) = 1 \times (-3) = -3$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\log_e\left(1 + \frac{x}{2}\right)}{x}, \quad \text{Ex : } \lim_{x \rightarrow 0} \frac{\log_e(1-2x)}{5x}$$

$$\text{Formula : } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$\text{Ex : } \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}}$$

$$\text{Ans : } \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x} \times 3} = \left(\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right)^3 = e^3$$

$$\text{Ex : } \lim_{x \rightarrow 0} (1+3x)^{\frac{2}{5x}}$$

$$\text{Ans : } \lim_{x \rightarrow 0} (1+3x)^{\frac{2}{5x}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x} \times \frac{2}{5}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x} \times 3 \times \frac{2}{5}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x} \times \frac{6}{5}}$$

$$= \left(\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right)^{\frac{6}{5}} = e^{\frac{6}{5}}$$

$$\text{Ex : } \lim_{x \rightarrow 0} (1+2x)^{\frac{1}{x}}, \quad \text{Ex : } \lim_{x \rightarrow 0} (1-2x)^{\frac{1}{5x}}$$

$$\text{Formula : } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 = 1 \times 3 = 3$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\sin 3x}{5x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 = \frac{1}{5} \times 3 = \frac{3}{5}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} = 1 \times \frac{1}{2}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 1^2 = 1$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{x^2}$$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right)^2 = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} = 1^2 \times \frac{1}{4} = \frac{1}{4}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right)^2 = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} \\ &= 2 \times 1^2 \times \frac{1}{4} = \frac{1}{2} \end{aligned}$$

OR

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x) \times (1 + \cos x)}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \times \lim_{x \rightarrow 0} \frac{1}{(1 + \cos x)} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times \frac{1}{2} = 1 \times \frac{1}{2} = \frac{1}{2}\end{aligned}$$

$$Ex: \lim_{x \rightarrow 0} \frac{\tan ax}{x}$$

$$Ans: \lim_{x \rightarrow 0} \frac{\tan ax}{x} = \lim_{x \rightarrow 0} \frac{\tan ax}{ax} \times a = 1 \times a = a \left(\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right)$$

$$Ex: \lim_{x \rightarrow 0} \frac{\sin^{-1} 4x}{x}$$

$$Ans: \lim_{x \rightarrow 0} \frac{\sin^{-1} 4x}{x} = \lim_{x \rightarrow 0} \frac{\sin^{-1} 4x}{4x} \times 4 = 1 \times 4 = 4 \left(\because \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \right)$$

$$Ex: \lim_{x \rightarrow 0} \frac{\tan^{-1} bx}{x}$$

$$Ans: \lim_{x \rightarrow 0} \frac{\tan^{-1} bx}{x} = \lim_{x \rightarrow 0} \frac{\tan^{-1} bx}{bx} \times b = 1 \times b = b \left(\because \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \right)$$

$$Ex: \lim_{x \rightarrow 0} \frac{x}{\sin 2x}$$

$$Ans: \lim_{x \rightarrow 0} \frac{x}{\sin 2x} = \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \times \frac{1}{2} = 1 \times \frac{1}{2} = \frac{1}{2} \left(\because \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \right)$$

$$Ex: \lim_{x \rightarrow 0} \frac{\sin mx}{\sin nx}$$

$$Ans: \lim_{x \rightarrow 0} \frac{\sin mx}{\sin nx} = \frac{\lim_{x \rightarrow 0} \frac{\sin mx}{x}}{\lim_{x \rightarrow 0} \frac{\sin nx}{x}} = \frac{\lim_{x \rightarrow 0} \frac{\sin mx}{mx} \times m}{\lim_{x \rightarrow 0} \frac{\sin nx}{nx} \times n} = \frac{1 \times m}{1 \times n} = \frac{m}{n}$$

$$Ex: \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, Ex: \lim_{x \rightarrow 0} \frac{\tan 3x}{\tan 5x}, Ex: \lim_{x \rightarrow 0} \frac{\tan 3x}{5x}, Ex: \lim_{x \rightarrow 0} \frac{\sin 2x}{7x}$$

Some problems by using substitution: $Ex: \lim_{h \rightarrow 0} \frac{(x+h)^5 - x^5}{h}$

$$Ans: \lim_{h \rightarrow 0} \frac{(x+h)^5 - x^5}{h} \left\{ \begin{array}{l} \text{Let } x+h = y \Rightarrow h = y-x \\ \text{As } h \rightarrow 0 \Rightarrow y \rightarrow x \end{array} \right.$$

$$= \lim_{y \rightarrow x} \frac{y^5 - x^5}{y-x} = 5x^{5-1} = 5x^4$$

$$Ex: \lim_{x \rightarrow 1} \frac{\ln x}{x-1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{\ln x}{x-1} \begin{cases} \text{Let } x-1 = y \Rightarrow x = 1+y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{\ln(1+y)}{y} = 1$$

$$Ex: \lim_{x \rightarrow 1} \frac{\ln(2x-1)}{x-1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{\ln(2x-1)}{x-1} \begin{cases} \text{Let } x-1 = y \Rightarrow x = 1+y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{\ln(2(1+y)-1)}{y} = \lim_{y \rightarrow 0} \frac{\ln(2+2y-1)}{y} = \lim_{y \rightarrow 0} \frac{\ln(1+2y)}{2y} \times 2 = 1 \times 2 = 2$$

$$Ex: \lim_{x \rightarrow 1} \frac{\ln x}{\sqrt{x}-1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{\ln x}{\sqrt{x}-1} = \lim_{x \rightarrow 1} \frac{\ln x}{(\sqrt{x}-1)(\sqrt{x}+1)} \times (\sqrt{x}+1) = \lim_{x \rightarrow 1} \frac{\ln x}{(x-1)} \times \lim_{x \rightarrow 1} (\sqrt{x}+1) \begin{cases} \text{Let } x-1 = y \Rightarrow x = 1+y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{\ln(1+y)}{y} \times (1+1) = 1 \times 2 = 2$$

$$Ex: \lim_{x \rightarrow 2} \frac{\ln(x-1)}{x-2}, Ex: \lim_{x \rightarrow 0} \frac{\ln(1+x)}{\sqrt{x+1}-1}, Ex: \lim_{x \rightarrow 2} \frac{\ln(x-1)}{(x-2)(x-3)}$$

$$Ex: \lim_{x \rightarrow 0} \frac{(x+9)^{\frac{3}{2}} - 27}{x}$$

$$Ex: \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{x-1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{x-1} \begin{cases} \text{Let } x-1 = y \Rightarrow x = 1+y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{3^y - 1}{y} = \ln 3$$

$$Ex: \lim_{x \rightarrow 1} \frac{3^x - 3}{x - 1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{3^x - 3}{x - 1} \begin{cases} \text{Let } x - 1 = y \Rightarrow x = 1 + y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{3^{y+1} - 3}{y} = \lim_{y \rightarrow 0} \frac{3 \times 3^y - 3}{y} = 3 \lim_{y \rightarrow 0} \frac{3^y - 1}{y} = 3 \ln 3 = \ln 3^3 = \ln 27$$

$$Ex: \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{\sqrt{x} - 1}$$

$$Ans: \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{\sqrt{x} - 1} = \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} \times (\sqrt{x} + 1) = \lim_{x \rightarrow 1} \frac{3^{x-1} - 1}{(x-1)} \times \lim_{x \rightarrow 1} (\sqrt{x} + 1) \begin{cases} \text{Let } x - 1 = y \Rightarrow x = 1 + y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{3^y - 1}{y} \times 2 = 2 \ln 3 = \ln 3^2 = \ln 9$$

$$Ex: \lim_{x \rightarrow 1} \frac{2^{x-1} - 1}{x - 1}, Ex: \lim_{x \rightarrow 1} \frac{2^{x-1} - 1}{\sqrt{x} - 1}$$

$$Ex: \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

$$Ans: \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} \begin{cases} \text{Let } x - \pi = y \Rightarrow x = \pi + y \\ \text{As } x \rightarrow \pi \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{\sin(\pi + y)}{-y} = \lim_{y \rightarrow 0} \frac{-\sin y}{-y} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$Ex: \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$$

$$Ans: \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} \begin{cases} \text{Let } x - \frac{\pi}{2} = y \Rightarrow x = \frac{\pi}{2} + y \\ \text{As } x \rightarrow \frac{\pi}{2} \Rightarrow y \rightarrow 0 \end{cases}$$

$$= \lim_{y \rightarrow 0} \frac{1 - \sin\left(\frac{\pi}{2} + y\right)}{(-y)^2} = \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2} = 2 \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{y} \right)^2$$

$$= 2 \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \times \frac{1}{2} \right)^2 = 2 \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} = 2 \times 1^2 \times \frac{1}{4} = \frac{1}{2}$$

$$Ex: \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \tan x$$

$$Ex: \lim_{x \rightarrow 1} \frac{x-1}{\ln x}$$

$$Ex: \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3}$$

$$\begin{aligned} Ans: \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} &= \lim_{x \rightarrow 0} \frac{2 \sin x - 2 \sin x \cos x}{x^3} = \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)}{x^3} \\ &= 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \lim_{x \rightarrow 0} \frac{(1 - \cos x)}{x^2} = 2 \times 1 \times \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = 2 \times 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right)^2 = 4 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 \\ &= 4 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} = 4 \times 1 \times \frac{1}{4} = 1 \end{aligned}$$

$$Ex: \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$$

$$\begin{aligned} Ans: \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{\sin x - \sin x \cos x}{\cos x}}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x (1 - \cos x)}{x^3 \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 \times \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} \times 1 = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right)^2 = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} = 2 \times 1 \times \frac{1}{4} = \frac{1}{2} \end{aligned}$$

$$Ex: \lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x \sin 2x}, Ex: \lim_{x \rightarrow 0} \frac{(1 - \cos x)^2}{\tan^3 x - \sin^3 x}, Ex: \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{\tan x}$$

$$Ex: \text{Find the value of } a \text{ If } \lim_{x \rightarrow \alpha} \frac{\tan a(x - \alpha)}{x - \alpha} = \frac{2}{3}$$

$$Ans: \lim_{x \rightarrow \alpha} \frac{\tan a(x - \alpha)}{x - \alpha} = \frac{2}{3} \begin{cases} \text{Let } x - \alpha = y \Rightarrow x = \alpha + y \\ \text{As } x \rightarrow \alpha \Rightarrow y \rightarrow 0 \end{cases}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\tan ay}{y} = \frac{2}{3}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\tan ay}{ay} \times a = \frac{2}{3} \Rightarrow 1 \times a = \frac{2}{3} \Rightarrow a = \frac{2}{3}$$

Ex: Find the value of a If $\lim_{x \rightarrow 1} \frac{5^x - 5}{(x-1) \ln a} = 5$

Ans: $\lim_{x \rightarrow 1} \frac{5^x - 5}{(x-1) \ln a} = 5 \begin{cases} \text{Let } x-1 = y \Rightarrow x = 1+y \\ \text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0 \end{cases}$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{5^{1+y} - 5}{y \ln a} = 5 \Rightarrow \lim_{y \rightarrow 0} \frac{5 \times 5^y - 5}{y \ln a} = 5 \Rightarrow \frac{5}{\ln a} \lim_{y \rightarrow 0} \frac{5^y - 1}{y} = 5 \Rightarrow \frac{1}{\ln a} \ln 5 = 1$$

$$\Rightarrow \ln 5 = \ln a \Rightarrow a = 5$$

Ex: Ex - 46: Find the value of a If $\lim_{x \rightarrow 2} \frac{\ln(2x-1)}{a(x-2)} = 2$

Ans: $\lim_{x \rightarrow 2} \frac{\ln(2x-1)}{a(x-2)} = 2 \begin{cases} \text{Let } x-2 = y \Rightarrow x = 2+y \\ \text{As } x \rightarrow 2 \Rightarrow y \rightarrow 0 \end{cases}$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\ln\{2(1+y)-1\}}{ay} = 2 \Rightarrow \lim_{y \rightarrow 0} \frac{\ln\{2+2y-1\}}{ay} = 2 \Rightarrow \frac{1}{a} \lim_{y \rightarrow 0} \frac{\ln(1+2y)}{y} = 2$$

$$\Rightarrow \frac{1}{a} \lim_{y \rightarrow 0} \frac{\ln(1+2y)}{2y} \times 2 = 2 \Rightarrow \frac{1}{a} \times 1 \times 2 = 2 \Rightarrow \frac{2}{a} = 2 \Rightarrow a = 1$$

Find the value of a

$$\text{Ex: } \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan ax} = \frac{1}{3}, \text{ Ex: } \lim_{x \rightarrow 0} \frac{e^{ax} - e^{-ax}}{x} = 3,$$

Limits at infinity or infinite limit

Meaning of $x \rightarrow \infty$: The symbol $x \rightarrow \infty$ will be used to mean that x takes very large values.

Ex: Show that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

Ans: As x takes very large value, $\frac{1}{x}$ becomes very small value, (i.e close to zero)

Thus, when $x \rightarrow \infty$, then $\frac{1}{x} \rightarrow 0$

Hence $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

Note:

$$1. \lim_{x \rightarrow \infty} \frac{3}{x} = 0$$

$$2. \lim_{x \rightarrow \infty} \frac{3}{x^2} = 0 \text{ and so on } \dots$$

Working rule for finding $\lim_{x \rightarrow \infty} f(x)$ Replace x by $\frac{1}{y}$ in the given function and take the limit as $y \rightarrow 0$

OR

In case of a rational function divide the numerator(N^r) and the denominator(D^r) by the highest power of x .

$$Ex: \lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 6}{2x^2 - 5x + 1}$$

$$Ans: \lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 6}{2x^2 - 5x + 1} = \lim_{x \rightarrow \infty} \frac{5 \frac{x^2}{x^2} + 3 \frac{x}{x^2} - \frac{6}{x^2}}{2 \frac{x^2}{x^2} - 5 \frac{x}{x^2} + \frac{1}{x^2}} \quad (\text{Divide the } N^r \text{ and } D^r \text{ by } x^2)$$

$$= \lim_{x \rightarrow \infty} \frac{5 + \frac{3}{x} - \frac{6}{x^2}}{2 - \frac{5}{x} + \frac{1}{x^2}} = \frac{5 + 0 - 0}{2 - 0 + 0} \quad (\text{put } x = \infty)$$

$$= \frac{5}{2}$$

$$Ex: \lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 5}$$

$$Ans: \lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 5} = \lim_{x \rightarrow \infty} \frac{2 \frac{x}{x} - \frac{3}{x}}{4 \frac{x}{x} + \frac{5}{x}} \quad (\text{Divide } x \text{ in both } N^r \text{ \& } D^r)$$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{3}{x}}{4 + \frac{5}{x}} = \frac{2 - 0}{4 + 0} = \frac{2}{4} = \frac{1}{2}$$

$$Ex: \lim_{n \rightarrow \infty} \frac{2n-3}{4n+5}, Ex: \lim_{x \rightarrow \infty} \frac{3x^3-4x^2+2}{4x^3+5x-1}, Ex: \lim_{x \rightarrow \infty} \frac{3n^2-4n+1}{4n^2+5n-2}, Ex: \lim_{n \rightarrow \infty} \frac{2n}{4n+5}$$

$$Ex: \lim_{x \rightarrow \infty} \frac{3x^3-4x^2+2}{4x^2+5x-3}$$

$$Ans: \lim_{x \rightarrow \infty} \frac{3x^3-4x^2+2}{4x^2+5x-3} = \lim_{x \rightarrow \infty} \frac{3\frac{x^3}{x^3}-4\frac{x^2}{x^3}+\frac{2}{x^3}}{4\frac{x^2}{x^3}+5\frac{x}{x^3}-\frac{3}{x^3}} \quad (\text{Divide } x^3 \text{ in both } N^r \text{ \& } D^r)$$

$$= \lim_{x \rightarrow \infty} \frac{3-\frac{4}{x}+\frac{2}{x^3}}{\frac{4}{x}+\frac{5}{x^2}-\frac{3}{x^3}} = \frac{3-0+0}{0+0-0} = \frac{3}{0} = \infty$$

$$Ex: \lim_{x \rightarrow \infty} \frac{3x^2-4x+2}{4x^3+5x^2-3}$$

$$Ans: \lim_{x \rightarrow \infty} \frac{3x^2-4x+2}{4x^3+5x^2-3} = \lim_{x \rightarrow \infty} \frac{3\frac{x^2}{x^3}-4\frac{x}{x^3}+\frac{2}{x^3}}{4\frac{x^3}{x^3}+5\frac{x^2}{x^3}-\frac{3}{x^3}} \quad (\text{Divide } x^3 \text{ in both } N^r \text{ \& } D^r)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{3}{x}-\frac{4}{x^2}+\frac{2}{x^3}}{4+\frac{5}{x}-\frac{3}{x^3}} = \frac{0-0+0}{4+0-0} = 0$$

$$Ex: \lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2}$$

$$Ans: \lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} \quad (\text{Divide } n \text{ in both } N^r \text{ \& } D^r)$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$Ex: \lim_{n \rightarrow \infty} \frac{1^3+2^3+3^3+\dots+n^3}{n^4}$$

$$Ans: \lim_{n \rightarrow \infty} \frac{1^3+2^3+3^3+\dots+n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} = \lim_{n \rightarrow \infty} \frac{\{n(n+1)\}^2}{4n^4} = \lim_{n \rightarrow \infty} \frac{n^2(n+1)^2}{4n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^2} + 2\frac{n}{n^2} + \frac{1}{n^2}}{4\frac{n^2}{n^2}} \quad (\text{Divide } n^2 \text{ in both } N^r \text{ \& } D^r)$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4} = \frac{1 + 0 + 0}{4} = \frac{1}{4}$$

$$\text{Ex: } \lim_{n \rightarrow \infty} 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$$

$$\text{Ans: } \lim_{n \rightarrow \infty} 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$$

$$= \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 2 \lim_{n \rightarrow \infty} \left\{ 1 - \left(\frac{1}{2}\right)^{n+1} \right\} = 2(1-0) \left(\lim_{n \rightarrow \infty} x^n = 0 \text{ if } -1 < x < 1 \right)$$

$$= 2$$

$$\text{Ex: } \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}}$$

$$\text{Ans: } \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}} = \frac{\lim_{n \rightarrow \infty} 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}}{\lim_{n \rightarrow \infty} 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}}$$

$$= \frac{\lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}}{\lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}}} = \frac{2 \lim_{n \rightarrow \infty} \left\{ 1 - \left(\frac{1}{2}\right)^{n+1} \right\}}{\frac{3}{2} \lim_{n \rightarrow \infty} \left\{ 1 - \left(\frac{1}{3}\right)^{n+1} \right\}} = \frac{4(1-0)}{3(1-0)} = \frac{4}{3}$$

$$\text{Ex: } \lim_{n \rightarrow \infty} 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}$$

Test the existence of the limit $\lim_{x \rightarrow a} f(x)$

The limit of a function $f(x)$ is said to exist at $x=a$ (i.e. $\lim_{x \rightarrow a} f(x)$ exist)

If Left hand limit (LHL) = Right hand limit (RHL)

Left hand limit (LHL)

$$\lim_{h \rightarrow 0} f(a-h) \text{ where } h \text{ is small (+ve) quantity}$$

$$\text{or } \lim_{x \rightarrow a^-} f(x)$$

Right hand limit (RHL)

$$\lim_{h \rightarrow 0} f(a+h) \text{ where } h \text{ is small (+ve) quantity}$$

$$\text{or } \lim_{x \rightarrow a^+} f(x)$$

Note: If L.H.L \neq R.H.L then the limit of a function does not exist

Ex: Test the existence of the limit $\lim_{x \rightarrow 1} |x-1|$

$$\text{Ans } \lim_{x \rightarrow 1} |x-1|$$

$$\text{Here } a = 1, f(x) = |x-1|$$

$$\begin{aligned} \text{L.H.L } \lim_{h \rightarrow 0} f(a-h) &= \lim_{h \rightarrow 0} f(\underline{1-h}) \text{ (put } x = 1-h \text{ in the given function } f(x) = |x-1|) \\ &= \lim_{h \rightarrow 0} |1-h-1| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\begin{aligned} \text{R.H.L } \lim_{h \rightarrow 0} f(a+h) &= \lim_{h \rightarrow 0} f(\underline{1+h}) \text{ (put } x = 1+h \text{ in the given function } f(x) = |x-1|) \\ &= \lim_{h \rightarrow 0} |1+h-1| = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\text{So L.H.L} = \text{R.H.L}$$

Hence the limit of the function exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow 0} |x|$

$$\text{Ans } \lim_{x \rightarrow 0} |x|$$

$$\text{Here } a = 0, f(x) = |x|$$

$$\begin{aligned} \text{L.H.L } \lim_{h \rightarrow 0} f(a-h) &= \lim_{h \rightarrow 0} f(\underline{0-h}) \text{ (put } x = 0-h \text{ in the given function } f(x) = |x|) \\ &= \lim_{h \rightarrow 0} |0-h| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\begin{aligned} \text{R.H.L } \lim_{h \rightarrow 0} f(a+h) &= \lim_{h \rightarrow 0} f(\underline{0+h}) \text{ (put } x = 0+h \text{ in the given function } f(x) = |x|) \\ &= \lim_{h \rightarrow 0} |0+h| = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\text{So L.H.L} = \text{R.H.L}$$

Hence the limit of the function exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow 0} \frac{|x|}{x}$

Ans $\lim_{x \rightarrow 0} \frac{|x|}{x}$

Here $a = 0, f(x) = \frac{|x|}{x}$

$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(0-h) \left(\text{put } x = 0-h \text{ in the given function } f(x) = \frac{|x|}{x} \right)$

$= \lim_{h \rightarrow 0} \frac{|0-h|}{0-h} = \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = \lim_{h \rightarrow 0} -1 = -1$

$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(0+h) \left(\text{put } x = 0+h \text{ in the given function } f(x) = \frac{|x|}{x} \right)$

$= \lim_{h \rightarrow 0} \frac{|0+h|}{0+h} = \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1$

So $L.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex: Test the existence of the limit

(a) $\lim_{x \rightarrow 2} |x-2|$ (b) $\lim_{x \rightarrow -1} |x+1|$ (c) $\lim_{x \rightarrow 0} \frac{x}{|x|}$ (d) $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$ (e) $\lim_{x \rightarrow 1} \frac{|x-2|}{x-2}$

Ex: Test the existence of the limit $\lim_{x \rightarrow 2} [x]$

Ans : $\lim_{x \rightarrow 2} [x]$

Here $a = 2, f(x) = [x]$

$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} [2-h] = \lim_{h \rightarrow 0} 1 = 1$

$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} [2+h] = \lim_{h \rightarrow 0} 2 = 2$

So $L.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow 0} [x]$

$$\text{Ans : } \lim_{x \rightarrow 0} [x]$$

Here $a = 0, f(x) = [x]$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} [0-h] = \lim_{h \rightarrow 0} -1 = -1$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} [0+h] = \lim_{h \rightarrow 0} 0 = 0$$

$SoL.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow 1} [x+1]$

$$\text{Ans : } \lim_{x \rightarrow 1} [x+1]$$

Here $a = 1, f(x) = [x+1]$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} [1-h+1] = \lim_{h \rightarrow 0} [2-h] = 1$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} [1+h+1] = \lim_{h \rightarrow 0} [2+h] = 2$$

$SoL.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow \sqrt{2}} [x]$

$$\text{Ans : } \lim_{x \rightarrow \sqrt{2}} [x]$$

Here $a = \sqrt{2}, f(x) = [x]$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(\sqrt{2}-h) = \lim_{h \rightarrow 0} [\sqrt{2}-h] = \lim_{h \rightarrow 0} 1 = 1$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(\sqrt{2}+h) = \lim_{h \rightarrow 0} [\sqrt{2}+h] = \lim_{h \rightarrow 0} 1 = 1$$

$SoL.H.L = R.H.L$

Hence the limit of the function exist.

Ex: Test the existence of the limit

$$(a) \lim_{x \rightarrow 1} [x] (b) \lim_{x \rightarrow -2} [x] (c) \lim_{x \rightarrow n} [x] \text{ if } n \text{ be an integer } (d) \lim_{x \rightarrow \sqrt{3}} [x]$$

Ex: Test the existence of the limit $\lim_{x \rightarrow 1} f(x)$

$$\text{If } f(x) = \begin{cases} 2x+1, & x < 1 \\ x-3, & x \geq 1 \end{cases}$$

Ans :

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(1-h) \quad \{ \text{Here } f(x) = 2x+1 \text{ and put } x = 1-h \}$$

$$= \lim_{h \rightarrow 0} \{2(1-h)+1\} = \lim_{h \rightarrow 0} \{2-2h+1\} = \lim_{h \rightarrow 0} 3-2h = 3$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(1+h) \quad \{ \text{Here } f(x) = x-3 \text{ and put } x = 1+h \}$$

$$= \lim_{h \rightarrow 0} \{(1+h) - 3\} = \lim_{h \rightarrow 0} (h-2) = -2$$

So $L.H.L \neq R.H.L$

Hence the limit does not exist.

Ex: Test the existence of the limit $\lim_{x \rightarrow 2} f(x)$

$$f(x) = \begin{cases} 2x-3, & x \leq 2 \\ x^2-3, & x > 2 \end{cases}$$

Ex: Test the existence of the limit $\lim_{x \rightarrow 2} f(x)$

$$f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

CONTINUITY

A function $f(x)$ is said to be continuous at a point $x=a$ if

$$\left\{ \begin{array}{l} \lim_{x \rightarrow a} f(x) = f(a) \\ \text{limiting value} = \text{functional value} \end{array} \right\}$$

OR

$$\left\{ \begin{array}{l} L.H.L = R.H.L = f(a) \\ \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(a+h) = f(a) \end{array} \right\}$$

DISCONTINUITY

A function $f(x)$ is said to be discontinuous at a point $x=a$ if

1. $f(a)$ is not defined
2. $\lim_{x \rightarrow a} f(x) \neq f(a)$
3. $L.H.L = R.H.L \neq f(a)$
4. $L.H.L \neq R.H.L = f(a)$
5. $L.H.L = f(a) \neq R.H.L$
6. $L.H.L \neq R.H.L \neq f(a)$
7. $\lim_{x \rightarrow a} f(x)$ does not exist

Ex: Test the continuity of the function

$$f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases} \quad \text{at } x = 0.$$

Ans: At $x = 0$, $f(x) = 3 \Rightarrow f(0) = 3$

$$\text{at } x \neq 0, f(x) = \frac{\sin 3x}{x}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 = 1 \times 3 = 3$$

$$\text{So } \lim_{x \rightarrow 0} f(x) = f(0) = 3$$

Hence the function is continuous at $x = 0$.

$$\text{Ex: Find the value of } a \text{ if the function } f(x) = \begin{cases} \frac{\sin ax}{2x}, & x \neq 0 \\ 3, & x = 0 \end{cases} \quad \text{is continuous at } x = 0.$$

Ans: At $x = 0$, $f(x) = 3 \Rightarrow f(0) = 3$

$$\text{at } x \neq 0, f(x) = \frac{\sin ax}{2x}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin ax}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin ax}{x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times a = \frac{1}{2} \times 1 \times a = \frac{a}{2}$$

Since the function is continuous at $x = 0$.

$$\text{So } \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\Rightarrow \frac{a}{2} = 3 \Rightarrow a = 6$$

Ex: Test the continuity of the function

$$f(x) = \begin{cases} \frac{x^7 - 1}{x - 1}, & x \neq 1 \\ 7, & x = 1 \end{cases} \quad \text{at } x = 1.$$

Ans: At $x = 1$, $f(x) = 7 \Rightarrow f(1) = 7$

$$\text{at } x \neq 1, f(x) = \frac{x^7 - 1}{x - 1}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^7 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{x^7 - 1^7}{x - 1} = 7 \times 1^6 = 7$$

$$\text{So } \lim_{x \rightarrow 1} f(x) = f(1) = 7$$

Hence the function is continuous at $x = 1$.

Ex: Test the continuity of the function

$$f(x) = \begin{cases} (1+3x)^{\frac{1}{x}}, & x \neq 0 \\ e^3, & x = 0 \end{cases} \quad \text{at } x = 0.$$

Ans : Ans : At $x = 0$, $f(x) = e^3 \Rightarrow f(0) = e^3$

at $x \neq 0$, $f(x) = (1+3x)^{\frac{1}{x}}$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x} \times 3} = \left\{ \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right\}^3 = e^3$$

So $\lim_{x \rightarrow 0} f(x) = f(0) = e^3$

Hence the function is continuous at $x = 0$.

Ex: Test the continuity of the function

$$(a) f(x) = \begin{cases} \frac{\sin 2x}{\tan 3x}, & x \neq 0 \\ \frac{2}{3}, & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(b) f(x) = \begin{cases} \frac{x^2 - a^2}{x - a}, & x \neq a \\ 2a, & x = a \end{cases} \quad \text{at } x = a$$

$$(c) f(x) = \begin{cases} \frac{e^{5x} - e^{-2x}}{x}, & x \neq 0 \\ 5, & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(d) f(x) = \begin{cases} \frac{a^x - b^x}{x}, & x \neq 0 \\ \ln \frac{a}{b}, & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(e) f(x) = \begin{cases} \frac{\ln(1+3x)}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases} \quad \text{at } x = 0$$

$$(f) f(x) = \begin{cases} (1+2x)^{\frac{1}{x}}, & x \neq 0 \\ e^2, & x = 0 \end{cases} \quad \text{at } x = 0$$

Ex: Test the continuity of the function

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases} \quad \text{at } x = 0$$

Ans: at $x = 0, f(x) = 0 \Rightarrow f(0) = 0$

at $x \neq 0, f(x) = \frac{|x|}{x}$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{|0-h|}{0-h} = \lim_{h \rightarrow 0} \frac{-h}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{|0+h|}{0+h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

So $L.H.L \neq R.H.L \neq f(0)$

Hence the function is discontinuous.

Ex: Show that the function $f(x) = [x]$ is not continuous at $x = 2$.

Ans: At $x = 2, f(x) = [x]$

$$\Rightarrow f(2) = [2] = 2$$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} [2-h] = \lim_{h \rightarrow 0} 1 = 1$$

$$R.H.L \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} [2+h] = \lim_{h \rightarrow 0} 2 = 2$$

So $L.H.L \neq R.H.L = f(0)$

Hence the function is discontinuous.

Ex: Test the continuity of the function

$$f(x) = \begin{cases} 2x+3, & x < 1 \\ x-4, & x \geq 1 \end{cases} \quad \text{at } x = 1$$

Ans: at $x = 1, f(x) = x - 4 \Rightarrow f(1) = 1 - 4 = -3$

$$L.H.L \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(1-h), \quad f(x) = 2x+3, \text{ (put } x = 1-h \text{ in the function } f(x) = 2x+3)$$

$$= \lim_{h \rightarrow 0} \{2(1-h) + 3\} = \lim_{h \rightarrow 0} 2 - 2h + 3 = 5$$

$$= \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(1+h), \quad f(x) = x-4, \text{ (put } x = 1+h \text{ in the function } f(x) = x-4)$$

$$= \lim_{h \rightarrow 0} \{(1+h) - 4\} = \lim_{h \rightarrow 0} 1 - 4 = -3$$

$L.H.L \neq R.H.L = f(1)$

Hence the function is discontinuous.

Ex: Test the continuity of the function

$$f(x) = \begin{cases} 2x-1, & x \leq 1 \\ x+2, & x > 1 \end{cases} \text{ at } x=1$$

Ex: Test the continuity of the function

$$f(x) = \begin{cases} 2x-1, & x < 0 \\ 3, & x = 0 \\ x+2, & x > 0 \end{cases} \text{ at } x=0$$

Ex: Test the continuity of the function

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases} \text{ at } x=0$$

Ex: Find the value of a if the function

$$f(x) = \begin{cases} \frac{\sin ax}{\tan x}, & x \neq 0 \\ \frac{1}{a}, & x = 0 \end{cases} \text{ is continuous at } x=0$$

Ex: Find the value of a and b if the function

$$f(x) = \begin{cases} ax^2 + b, & x < 1 \\ 1, & x = 1 \\ 2ax - b, & x > 1 \end{cases} \text{ is continuous at } x=1$$

(hints-see the semester question below)

Formula: $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n$

proof: $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$ let $1+x = y \Rightarrow x = y-1$

As $x \rightarrow 0 \Rightarrow y \rightarrow 1$

$$= \lim_{y \rightarrow 1} \frac{y^n - 1^n}{y - 1} = n \times 1^{n-1} = n$$

Note: $\lim_{x \rightarrow 0} \frac{(1-x)^n - 1}{x} = -n$

Ex: $\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{2}} - 1}{x} = \frac{1}{2}$ (by using formula)

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{2}} - 1}{x} = -\frac{1}{2} \text{ (by using formula)}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{3}{2}} - 1}{x} = \frac{3}{2} \text{ (by using formula)}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 + 1 - \sqrt{1-x}}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1) - (\sqrt{1-x} - 1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)}{x} - \lim_{x \rightarrow 0} \frac{(\sqrt{1-x} - 1)}{x} = \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{2}} - 1}{x} - \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{2}} - 1}{x} \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

$$\text{Ex : } \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1 + 1 - \sqrt[3]{1-x}}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt[3]{1+x} - 1) - (\sqrt[3]{1-x} - 1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt[3]{1+x} - 1)}{x} - \lim_{x \rightarrow 0} \frac{(\sqrt[3]{1-x} - 1)}{x} = \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{3}} - 1}{x} - \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{3}} - 1}{x} \\ &= \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

$$\text{Ex : } \frac{\lim_{x \rightarrow 0} \sqrt{1+x} - \sqrt{1-x}}{\lim_{x \rightarrow 0} \sqrt[3]{1+x} - \sqrt[3]{1-x}}$$

$$\begin{aligned} \text{Ans : Ex : } \frac{\lim_{x \rightarrow 0} \sqrt{1+x} - \sqrt{1-x}}{\lim_{x \rightarrow 0} \sqrt[3]{1+x} - \sqrt[3]{1-x}} &= \frac{\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}}{\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}} = \frac{\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 + 1 - \sqrt{1-x}}{x}}{\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1 + 1 - \sqrt[3]{1-x}}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1) - (\sqrt{1-x} - 1)}{x}}{\lim_{x \rightarrow 0} \frac{(\sqrt[3]{1+x} - 1) - (\sqrt[3]{1-x} - 1)}{x}} = \frac{\lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)}{x} - \lim_{x \rightarrow 0} \frac{(\sqrt{1-x} - 1)}{x}}{\lim_{x \rightarrow 0} \frac{(\sqrt[3]{1+x} - 1)}{x} - \lim_{x \rightarrow 0} \frac{(\sqrt[3]{1-x} - 1)}{x}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{2}} - 1}{x} - \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{2}} - 1}{x}}{\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{3}} - 1}{x} - \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{3}} - 1}{x}} = \frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{\frac{1}{3} - \left(-\frac{1}{3}\right)} = \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}
 \end{aligned}$$

SEMESTER QUESTIONS

Q.1 Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos^2 x}{1 - \sin x} \right)$

$$\begin{aligned}
 \text{Ans : } \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos^2 x}{1 - \sin x} \right) &= \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1 - \sin^2 x}{1 - \sin x} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x} \right) \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} (1 + \sin x) = 1 + \sin \frac{\pi}{2} = 1 + 1 = 2
 \end{aligned}$$

Q.2 Evaluate $\lim_{x \rightarrow 0} \left(\frac{\cos ecx - \cot x}{x} \right)$

$$\begin{aligned}
 \text{Ans : } \lim_{x \rightarrow 0} \left(\frac{\cos ecx - \cot x}{x} \right) &= \lim_{x \rightarrow 0} \left(\frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x \sin x} \right) = \lim_{x \rightarrow 0} \left(\frac{2 \sin^2 \frac{x}{2}}{x \cdot 2 \sin \frac{x}{2} \cos \frac{x}{2}} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x \cos \frac{x}{2}} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right) \times \lim_{x \rightarrow 0} \left(\frac{1}{\cos \frac{x}{2}} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right) \times \frac{1}{2} \times 1 = 1 \times \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

Q.3 Discuss the continuity of the function

$$f(x) = \begin{cases} 3x - 2 & \text{when } x \leq 0 \\ x + 1 & \text{when } x > 0 \end{cases} \text{ at } x = 0$$

Ans:

$$\text{At } x = 0, f(x) = 3x - 2 \Rightarrow f(0) = 3 \times 0 - 2 = -2$$

L.H.L

$$\lim_{h \rightarrow 0} f(a - h) = \lim_{h \rightarrow 0} f(0 - h), f(x) = 3x - 2$$

$$= \lim_{h \rightarrow 0} \{3(0 - h) - 2\} = -2,$$

R.H.L

$$\begin{aligned}\lim_{h \rightarrow 0} f(a+h) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} (0+h+1) = 1, \quad f(x) = x+1\end{aligned}$$

So $L.H.L = f(0) \neq R.H.L$

Hence the function is discontinuous.

Q.4 Evaluate $\lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x})$

$$\begin{aligned}\text{Ans: } \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x}) &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{(\sqrt{x+1} + \sqrt{x})} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}(x+1-x)}{(\sqrt{x+1} + \sqrt{x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+\frac{1}{x}} + 1\right)} = \lim_{x \rightarrow \infty} \frac{1}{\left(\sqrt{1+\frac{1}{x}} + 1\right)} = \frac{1}{2}\end{aligned}$$

Q.5 Evaluate $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x}$

$$\text{Ans: } \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} \quad \text{let } \sin^{-1} x = y \Rightarrow x = \sin y$$

$$\text{As } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{y}{\sin y} = 1$$

Q.6 Evaluate $\lim_{x \rightarrow 1} \frac{\log_e(2x-1)}{x-1}$

Ans:

$$\lim_{x \rightarrow 1} \frac{\log_e(2x-1)}{x-1} \quad \text{let } x-1 = y \Rightarrow x = 1+y$$

$$\text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{\log_e(2(1+y)-1)}{y} = \lim_{y \rightarrow 0} \frac{\log_e(2+2y-1)}{y} = \lim_{y \rightarrow 0} \frac{\log_e(1+2y)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{\log_e(1+2y)}{2y} \times 2 = 1 \times 2 = 2$$

Q.7 If $f(x) = \begin{cases} ax^2 + b, & x < 1 \\ 1, & x = 1 \\ 2ax - b, & x > 1 \end{cases}$ is continuous at $x=1$, then find a and b .

Ans: at $x=1$, $f(x)=1 \Rightarrow f(1)=1$

L.H.L

$$\begin{aligned} \lim_{h \rightarrow 0} f(a-h) &= \lim_{h \rightarrow 0} f(1-h) \\ &= \lim_{h \rightarrow 0} a(1-h)^2 + b = a + b \end{aligned} \quad , f(x) = ax^2 + b$$

R.H.L

$$\begin{aligned} \lim_{h \rightarrow 0} f(a+h) &= \lim_{h \rightarrow 0} f(1+h) \\ &= \lim_{h \rightarrow 0} 2a(1+h) - b = 2a - b \end{aligned} \quad , f(x) = 2ax - b$$

Since the function is continuous, so L.H.L = R.H.L = $f(1)$

So

$$a + b = 1$$

$$\underline{2a - b = 1}$$

$$\Rightarrow 3a = 2 \Rightarrow a = \frac{2}{3} \quad \text{so } a + b = 1 \Rightarrow b = 1 - \frac{2}{3} = \frac{1}{3}$$

Q.8 Evaluate $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$

Ans: $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$ let $x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$

As $x \rightarrow \infty \Rightarrow y \rightarrow 0$

$$= \lim_{y \rightarrow 0} \frac{1}{y} \sin y = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Q.9 Evaluate $\lim_{x \rightarrow 0} \frac{(x+9)^{\frac{3}{2}} - 27}{x}$

Ans: $\lim_{x \rightarrow 0} \frac{(x+9)^{\frac{3}{2}} - 27}{x}$ Let $x+9 = y \Rightarrow x = y-9$

As $x \rightarrow 0 \Rightarrow y \rightarrow 9$

$$= \lim_{y \rightarrow 9} \frac{y^{\frac{3}{2}} - 9^{\frac{3}{2}}}{y - 9} = \frac{3}{2} \times 9^{\frac{3}{2}-1} = \frac{3}{2} \times 9^{\frac{1}{2}} = \frac{3}{2} \times 3 = \frac{9}{2}$$

Q.10 If $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ examine the continuity of $f(x)$ at $x=0$.

Ans: at $x=0, f(x)=1 \Rightarrow f(0)=1$

At $x \neq 0, f(x) = \frac{|x|}{x}$

L.H.L $\lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{|0-h|}{0-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = \lim_{h \rightarrow 0} -1 = -1$

R.H.L $\lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{|0+h|}{0+h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1$

LHL \neq RHL = $f(a)$

Hence the function is not continuous at $x=0$.

Q.11 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$

Ans: $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} = \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{x}}{\lim_{x \rightarrow 0} \frac{\sin 5x}{x}} = \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3}{\lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times 5} = \frac{1 \times 3}{1 \times 5} = \frac{3}{5}$

Q.12 Examine the continuity of the function $f(x)$ at $x=0$ defined by $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x = 0$.

Ans: At $x=0, f(x)=2 \Rightarrow f(0)=2$

At $x \neq 0, f(x) = \frac{\sin 2x}{x}$

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 = 1 \times 2 = 2$

So $\lim_{x \rightarrow 0} f(x) = f(0) = 2$

Hence the function is continuous.

Q.13 Show that $\lim_{x \rightarrow 0} \frac{x}{|x|}$ **does not exist.**

Ans: Here $a=0$, $f(x) = \frac{x}{|x|}$

$$\text{L.H.L } \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{0-h}{|0-h|} = \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} -1 = -1$$

$$\text{R.H.L } \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{0+h}{|0+h|} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1$$

So LHL \neq RHL, Hence the limit does not exist.

Q.14 If $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan ax} = 1$, find the value of a .

Ans:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan ax} = 1 &\Rightarrow \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{x}}{\lim_{x \rightarrow 0} \frac{\tan ax}{x}} = 1 \Rightarrow \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3}{\lim_{x \rightarrow 0} \frac{\tan ax}{ax} \times a} = 1 \\ &\Rightarrow \frac{1 \times 3}{1 \times a} = 1 \Rightarrow \frac{3}{a} = 1 \Rightarrow a = 3 \end{aligned}$$

Q.15 If $f(x) = \begin{cases} (1+3x)^{\frac{1}{3}}, & x \neq 0 \\ e^3, & x = 0 \end{cases}$ then examine the continuity of $f(x)$ at $x=0$.

Ans: At $x=0$, $f(x) = e^3 \Rightarrow f(0) = e^3$

$$\text{At } x=0, f(x) = (1+3x)^{\frac{1}{3}}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3}} = \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3} \times 3} = \left\{ \lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3}} \right\}^3 = e^3$$

So $\lim_{x \rightarrow 0} f(x) = f(0)$

Hence the function is continuous.

Q.16 Evaluate $\lim_{x \rightarrow 0} \frac{x - x \cos 2x}{\sin^3 2x}$

$$\begin{aligned} \text{Ans: } \lim_{x \rightarrow 0} \frac{x - x \cos 2x}{\sin^3 2x} &= \lim_{x \rightarrow 0} \frac{x(1 - \cos 2x)}{(\sin 2x)^3} = \lim_{x \rightarrow 0} \frac{x 2 \sin^2 x}{(2 \sin x \cos x)^3} = \lim_{x \rightarrow 0} \frac{x 2 \sin^2 x}{8 \sin^3 x \cos^3 x} \\ &= \lim_{x \rightarrow 0} \frac{x}{4 \sin x \cos^3 x} = \frac{1}{4} \lim_{x \rightarrow 0} \frac{x}{\sin x} \times \lim_{x \rightarrow 0} \frac{1}{\cos^3 x} = \frac{1}{4} \times 1 \times \frac{1}{1} = \frac{1}{4} \end{aligned}$$

Q.17 Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

$$\begin{aligned} \text{Ans: } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{x} \right)^2 = 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} = 2 \times 1^2 \times \frac{1}{4} = \frac{1}{2} \end{aligned}$$

Q.18 Determine the value of α such that $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ \alpha & , x = 3 \end{cases}$ is continuous at $x=3$.

Ans: At $x=3$, $f(x) = \alpha \Rightarrow f(3) = \alpha$

At $x \neq 3$, $f(x) = \frac{x^2 - 9}{x - 3}$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{x^2 - 3^2}{x - 3} = 2 \times 3^1 = 6$$

Since the function is continuous at $x=3$. So $\lim_{x \rightarrow 3} f(x) = f(3)$

$$\Rightarrow 6 = \alpha.$$

Q.19 Evaluate $\lim_{x \rightarrow 1} \frac{\ln(2x-1)}{x-1}$

$$\text{Ans: } \lim_{x \rightarrow 1} \frac{\ln(2x-1)}{x-1} \quad \text{Let } x-1 = y \Rightarrow x = 1+y$$

$$\text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{\ln(2(1+y)-1)}{y} = \lim_{y \rightarrow 0} \frac{\ln(2+2y-1)}{y} = \lim_{y \rightarrow 0} \frac{\ln(1+2y)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{\ln(1+2y)}{2y} \times 2 = 1 \times 2 = 2$$

Q.20 Evaluate $\lim_{x \rightarrow 0} \frac{\sin x^0}{x}$

$$\text{Ans: } \lim_{x \rightarrow 0} \frac{\sin x^0}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{\pi}{180} x}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{\pi}{180} x}{\frac{\pi}{180} x} \times \frac{\pi}{180} = 1 \times \frac{\pi}{180} = \frac{\pi}{180}$$

Q.21 Evaluate $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$

$$\begin{aligned} \text{Ans: } \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3} &= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 2n + n + 1}{6n^2} = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \frac{\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{6n^2}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{2 + 0 + 0}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Q.22 Examine the continuity of the function

$$f(x) \text{ at } x = 0 \text{ defined by } f(x) = \begin{cases} 2x+1 & \text{when } x < 1 \\ 0 & \text{when } x = 0 \\ x^2 - 1 & \text{when } x > 1 \end{cases}$$

$$\text{Ans: At } x = 0, f(x) = 0 \Rightarrow f(0) = 0$$

$$\begin{aligned} L.H.L \lim_{h \rightarrow 0} f(a-h) &= \lim_{h \rightarrow 0} f(0-h), f(x) = 2x+1 \\ &= \lim_{h \rightarrow 0} \{2(0-h) + 1\} = 1 \end{aligned}$$

$$\begin{aligned} R.H.L \lim_{h \rightarrow 0} f(a+h) &= \lim_{h \rightarrow 0} f(0+h), f(x) = x^2 - 1 \\ &= \lim_{h \rightarrow 0} \{(0-h)^2 - 1\} = -1 \end{aligned}$$

$$\text{So } L.H.L \neq R.H.L \neq f(0)$$

Hence the function is not continuous

Q.23

Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$\begin{aligned} \text{Ans : } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{x(\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{1+x-1-x}{x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2}{2} = 1 \end{aligned}$$

Q.24

Evaluate $\lim_{x \rightarrow 0} \frac{\tan 5x}{\tan 7x}$

$$\text{Ans : } \lim_{x \rightarrow 0} \frac{\tan 5x}{\tan 7x} = \lim_{x \rightarrow 0} \frac{\frac{\tan 5x}{x}}{\frac{\tan 7x}{x}} = \lim_{x \rightarrow 0} \frac{\frac{\tan 5x}{5x} \times 5}{\frac{\tan 7x}{7x} \times 7} = \frac{1 \times 5}{1 \times 7} = \frac{5}{7}$$

Q.25

Test the continuity of the function at $x = 0$ if

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Ans : at $x = 0$, $f(x) = 0 \Rightarrow f(0) = 0$

at $x \neq 0$, $f(x) = x \sin \frac{1}{x}$

We have $-1 \leq \sin \frac{1}{x} \leq 1 \Rightarrow -x \leq x \sin \frac{1}{x} \leq x$

$$\Rightarrow \lim_{x \rightarrow 0} -x \leq \lim_{x \rightarrow 0} x \sin \frac{1}{x} \leq \lim_{x \rightarrow 0} x$$

but $\lim_{x \rightarrow 0} -x = 0$ and $\lim_{x \rightarrow 0} x = 0$ by using sandwich theorem we get

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

So $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = f(0)$

Hence the function is continuous

Reference Books: Elements of Mathematics Vol. - 2 (Odisha State Bureau of Text Book preparation & Production)