KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -II

PART-2

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CONTENTS

CHAPTER-3

DERIVATIVE

Derivative is the rate of change of one quantity with respect to another quantity.

In mathematics we have to find the rate of change of one variable with respect to another variable.

Let us consider a function $y = f(x)$. Here x is an indepedent variable and y is the dependent variable

Here we have to find the rate of change of the dependent variable (y) with respect an indepedent variable (x). It is denoted by $\frac{dy}{dx}$ (which is read as derivative of y with respect to x).

Method for finding $\frac{dy}{dx}$ (Using definition or 1st principle or ab initio)

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(provided that the limit exist)}$$

Note: If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$

If $y = f(\theta)$ then $\frac{dy}{d\theta} = f'(\theta)$ (Here y is dependent variable and θ is an indepedent variable)

Note: Also $\frac{dy}{dx}$ is denoted by y' or Dy $\left(D = \frac{d}{dx}\right)$

Ex: Differentiate $y = x^n$ by using 1st principle or ab initio method.

Ans: Given $y = x^n = f(x)$ [suppose]

$$\text{We have } \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \quad \{\because f(x) = x^n\}$$

$$\left(\begin{array}{l} \text{Let } x+h = y \Rightarrow h = y - x \\ \text{As } h \to 0 \Rightarrow y \to x \end{array}\right)$$

$$= \lim_{y\to x} \frac{y^n - x^n}{y - x} = nx^{n-1} \quad \left[\because \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method.

$(a)\ y=x^9\left(\frac{dy}{dx}=9x^8\right)$ $(b)\ y=\frac{1}{x^5}$ or $y=x^{-5}\left(\frac{dy}{dx}=-5x^{-6}\right)$ $(c)\ y=\sqrt{x}$ or $y=x^{\frac{1}{2}}\left(\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\right)$

$(d)\ y=\frac{1}{x}$ or $y=x^{-1}\left(\frac{dy}{dx}=-\frac{1}{x^2}\right)$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=e^x$.

Ans: *Given* $y=e^x = f(x)$[suppose]

We have $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\quad \{\because f(x)=e^x\}$

$$=\lim_{h\to 0}\frac{e^xe^h-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}=e^x\lim_{h\to 0}\frac{(e^h-1)}{h}=e^x\times 1=e^x\left[\because \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=a^x$.

Ans: *Given* $y=a^x = f(x)$[suppose]

We have $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\quad \{\because f(x)=a^x\}$

$$=\lim_{h\to 0}\frac{a^xa^h-a^x}{h}=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}=a^x\lim_{h\to 0}\frac{(a^h-1)}{h}=a^x\times \ln a=a^x\ln a\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\ln a\right]$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=\ln x$.

Ans: *Given* $y=\ln x = f(x)$[suppose]

We have $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}\quad \{\because f(x)=\ln x\}$

$$=\lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}=\lim_{h\to 0}\frac{\ln\left(\frac{x}{x}+\frac{h}{x}\right)}{h}=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\times\frac{1}{x}$$

$$=1\times\frac{1}{x}=\frac{1}{x}\left[\because \lim_{x\to 0}\frac{\ln(1+x)}{x}=1\right]$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=\sin x$.

$Ans: Given\ y=\sin x = f(x)$[suppose]

$$We\ have\ \frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \quad \{\because f(x)=\sin x\}$$

$$=\lim_{h\to 0}\frac{2\cos\frac{x+h+x}{2}\times\sin\frac{x+h-x}{2}}{h}=2\lim_{h\to 0}\frac{\cos\frac{2x+h}{2}\times\sin\frac{h}{2}}{h}=2\lim_{h\to 0}\cos\frac{2x+h}{2}\times\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h}$$

$$=2\cos x\times\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\times\tfrac{1}{2}=2\cos x\times 1\times\tfrac{1}{2}=\cos x$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=\tan x$.

$Ans: Given\ y=\tan x = f(x)$[suppose]

$$We\ have\ \frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \quad \{\because f(x)=\tan x\}$$

$$=\lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}=\lim_{h\to 0}\frac{\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{\cos(x+h)\cos x}}{h}=\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{\sinh}{h}\times\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}=1\times\frac{1}{\cos x\times\cos x}=\frac{1}{\cos^2 x}=\sec^2 x$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method of $y=\sec x$.

$Ans: Given\ y=\sec x = f(x)$[suppose]

$$We\ have\ \frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h} \quad \{\because f(x)=\sec x\}$$

$$=\lim_{h\to 0}\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}=\lim_{h\to 0}\frac{\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}}{h}=\lim_{h\to 0}\frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{2\sin\frac{x+x+h}{2}\times\sin\frac{x+h-x}{2}}{h\cos(x+h)\cos x}=\lim_{h\to 0}\frac{2\sin\frac{2x+h}{2}\times\sin\frac{h}{2}}{h\cos(x+h)\cos x}$$

$$= 2\lim_{h\to 0}\sin\frac{2x+h}{2}\times\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h}\times\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x} = 2\sin x\times\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\times\frac{1}{2}\times\frac{1}{\cos x\cos x}$$

$$= 2\sin x\times 1\times\frac{1}{2}\times\frac{1}{\cos^2 x} = \sec x\tan x$$

Ex:Find $\frac{dy}{dx}$ by using 1st principle or ab initio method.

$(a)\ y=\cos x\left(\frac{dy}{dx}=-\sin x\right)(b)\ y=\cot x\left(\frac{dy}{dx}=-\operatorname{cosec}^2 x\right)(c)\ y=\operatorname{cosec} x\left(\frac{dy}{dx}=-\operatorname{cosec} x\cot x\right)$

FORMULAS

$F-1.\frac{d}{dx}(x^n)=nx^{n-1}$

$F-2.\frac{d}{dx}(e^x)=e^x$

$F-3.\frac{d}{dx}(a^x)=a^x\log_e a \text{ or } a^x\ln a\ (a>0)$

$F-4.\frac{d}{dx}(\ln x)=\frac{d}{dx}(\log_e x)=\frac{1}{x}$

$F-5.\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$

$F-6.\frac{d}{dx}(\sin x)=\cos x$

$F-8.\frac{d}{dx}(\cos x)=-\sin x$

$F-9.\frac{d}{dx}(\tan x)=\sec^2 x$

$F-10.\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

$F-11.\frac{d}{dx}(\sec x)=\sec x\tan x$

$F-12.\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$

$F-13.\frac{d}{dx}(x^2)=2x$

$F-14.\frac{d}{dx}(x)=1$

$F-15.\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2\sqrt{x}}$

$F-16.\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

$F-17.\frac{d}{dx}(k)=0 \text{ where k is a constant}$

$Formula\ \frac{d}{dx}\left(x^n\right)=nx^{n-1}$

$Ex:Find\ \frac{dy}{dx}\ if\ y=x^9$

$Ans: y=x^9$

$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^9\right)=9x^{9-1}=9x^8$

$Ex: Find\ \frac{dy}{dx}\ if\ y=x^{-6}=\frac{1}{x^6}$

$Ans: y=x^{-6}\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^{-6}\right)=-6x^{-6-1}=-6x^{-7}$

$Ex: Find\ \frac{dy}{dx}\ if\ y=x^{\frac{5}{3}}$

$Ans: y=x^{\frac{5}{3}}\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{5}{3}}\right)=\frac{5}{3}x^{\frac{5}{3}-1}=\frac{5}{3}x^{\frac{2}{3}}$

$Ex: Find\ \frac{dy}{dx}\ if\ y=x^{-\frac{5}{3}}$

$Ans: y=x^{-\frac{5}{3}}\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^{-\frac{5}{3}}\right)=-\frac{5}{3}x^{-\frac{5}{3}-1}=-\frac{5}{3}x^{\frac{-8}{3}}$

$Ex: Find\ \frac{dy}{dx}\ of\ (a)\ y=\sqrt[3]{x}=x^{\frac{1}{3}}\ (b)\ y=\frac{1}{x\sqrt{x}}\left(\text{Hints}\quad x\sqrt{x}=x^{\frac{3}{2}}\right)$

$Formula\ \frac{d}{dx}\left(a^x\right)=a^x\ln a$

$Ex: Find\ \frac{dy}{dx}\ if\ y=5^x$

$Ans: y=5^x\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(5^x\right)=5^x\ln 5$

$Ex: Find\ \frac{dy}{dx} if\ y = 3^x$

$Ans: y = 3^x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(3^x) = 3^x \ln 3$

$Ex: Find\ \frac{dy}{dx} if\ y = 7^x$

$Formula\ \ \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

$Ex: Ex: Find\ \frac{dy}{dx}\ if\ y = \log_5 x$

$Ans: y = \log_5 x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\log_5 x) = \frac{1}{x \ln 5}$

$Ex: Find\ \frac{dy}{dx} if\ y = \log_{10} x$

ALGEBRA OF DERIVATIVE

$F-1\frac{d}{dx}(U \pm V) = \frac{dU}{dx} \pm \frac{dV}{dx}$

$Ex: Find\ \frac{dy}{dx} if\ y = x^5 + \tan x - e^x + 7$

$Ans: y = x^5 + \tan x - e^x + 7$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^5 + \tan x - e^x + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(\tan x) - \frac{d}{dx}(e^x) + \frac{d}{dx}7$

$= 5x^4 + \sec^2 x - e^x$

$Ex: Find\ \frac{dy}{dx}(a)\ y = x^3 + e^x + 3^x - \cot x + 9\ (b)\ y = x^4 + \log_3 x - \sec x + \cos x + 3$

$F-2.\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$

$Ex: Find\ \frac{dy}{dx} if\ y = x^5 \tan x$

$Ans: y = x^5 \tan x$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^5 \tan x) = x^5 \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(x^5)$

$= x^5 \sec^2 x + \tan x \times 5x^4 = x^5 \sec^2 x + 5x^4 \tan x$

$Ex: Find\ \frac{dy}{dx}(a)\ y = x^p \cos x\ (b)\ y = e^x \sin x$

$$F-3.\frac{d}{dx}\left(\frac{U}{V}\right)=\frac{V\frac{dU}{dx}-U\frac{dV}{dx}}{V^2}$$

$Ex: Find\ \frac{dy}{dx} if\ y=\frac{x^5}{\cos x}$

$Ans: y=\frac{x^5}{\cos x}$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^5}{\cos x}\right)=\frac{\cos x\frac{d}{dx}\left(x^5\right)-x^5\frac{d}{dx}\left(\cos x\right)}{(\cos x)^2}=\frac{\cos x\times 5x^4-x^5\left(-\sin x\right)}{\cos^2 x}$$

$$=\frac{5x^4\cos x+x^5\sin x}{\cos^2 x}$$

$Ex: Find\ \frac{dy}{dx}\left(a\right)y=\frac{\tan x}{\ln x}\left(b\right)y=\frac{\log x}{x^3}$

$F-4.\frac{d}{dx}\left\{k\ f\left(x\right)\right\}=k\frac{d}{dx}\left\{f\left(x\right)\right\}$ where k is a constant

Ex:$Find\ \frac{dy}{dx} if\ y=6\tan x$

$Ans: y=6\tan x$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(6\tan x\right)=6\frac{d}{dx}\left(\tan x\right)=6\sec^2 x$$

$Ex: Find\ \frac{dy}{dx}\left(a\right)y=5\ln x\ \left(b\right)y=5x^4$

$Ex: Find\ \frac{dy}{dx} if\ y=5x^3+3\tan x-7\ln x+9$

$Ans: y=5x^3+3\tan x-7\ln x+9$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(5x^3\right)+\frac{d}{dx}\left(3\tan x\right)-\frac{d}{dx}\left(7\ln x\right)+\frac{d}{dx}9$$

$$=5\times 3x^2+3\sec^2 x-7\frac{1}{x}=15x^2+3\sec^2 x-\frac{7}{x}$$

$Ex: Find\ \frac{dy}{dx} if\ y=5x^3+3\tan x-7\ln x+9$

$Ans: y=5x^3+3\tan x-7\ln x+9$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(5x^3\right)+\frac{d}{dx}\left(3\tan x\right)-\frac{d}{dx}\left(7\ln x\right)+\frac{d}{dx}9$$

$$=5\times 3x^2+3\sec^2 x-7\frac{1}{x}=15x^2+3\sec^2 x-\frac{7}{x}$$

$Ex: Find\ \frac{dy}{dx} if\ y=\cot x+x^2\log_2 x$

$Ans: y=\cot x+x^2\log_2 x$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(\cot x\right)+\frac{d}{dx}\left(x^2\log_2 x\right)=-\cos ec^2x+x^2\frac{d}{dx}\left(\log_2 x\right)+\left(\log_2 x\right)\frac{d}{dx}x^2$$

$$=-\cos ec^2x+x^2\frac{1}{x\ln 2}+\left(\log_2 x\right)2x=-\cos ec^2x+\frac{x}{\ln 2}+2x\left(\log_2 x\right)$$

$Ex: Find\ \frac{dy}{dx} if\ y=\frac{x^3+4}{x^2-2}$

$Ans: y=\frac{x^3+4}{x^2-2}$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^3+4}{x^2-2}\right)=\frac{(x^2-2)\frac{d}{dx}\left(x^3+4\right)-\left(x^3+4\right)\frac{d}{dx}\left(x^2-2\right)}{\left(x^2-2\right)^2}$$

$$=\frac{(x^2-2)3x^2-\left(x^3+4\right)2x}{\left(x^2-2\right)^2}=\frac{3x^4-6x^2-2x^4-8x}{\left(x^2-2\right)^2}=\frac{x^4-6x^2-8x}{\left(x^2-2\right)^2}$$

$Ex: Find\ \frac{dy}{dx} if\ y=\frac{1-\cos x}{1+\cos x}$

$Ans: y=\frac{1-\cos x}{1+\cos x}$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)\left(-(-\sin x)\right)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}=\frac{(1+\cos x)\sin x+(1-\cos x)\sin x}{(1+\cos x)^2}$$

$$=\frac{\sin x+\sin x\cos x+\sin x-\sin x\cos x}{(1+\cos x)^2}=\frac{2\sin x}{(1+\cos x)^2}$$

$Ex: Find\ \frac{dy}{dx}(a)\ y=\frac{1-\sin x}{1+\sin x}\ \ (b)\ y=\frac{1-\tan x}{1+\tan x}\ \ (c)\ y=\frac{\sqrt{x}-1}{\sqrt{x}+1}(d)\ y=x\sin x-\frac{e^x}{x^2+1}$

$(e)\ y=\frac{3^x-2^x}{\ln x}$

DERIVATIVE OF COMPOSITE FUNCTION (CHAIN RULE)

Let us consider a composite function $y = f(g(x))$, so we have to find the derivative of y with respect to $x \left(i.e \frac{dy}{dx} \right)$

$Now\ y = f(g(x))$

$Let\ u = g(x)$

$So\ y = f(u) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(f(u)) = f'(u)$

$Again\ u = g(x) \Rightarrow \frac{du}{dx} = \frac{d}{dx}(g(x)) = g'(x)$

$\therefore \frac{dy}{dx} = \frac{dy}{dx} \times \frac{du}{dx} = f'(u)g'(x) = f'(g(x))g'(x)$

Inverse Trigonometric Function

$F-1. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

$F-2. \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

$F-3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, x \in R$

$F-4. \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, x \in R$

$F-5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, x \in R - [-1,1]$

$F-6. \frac{d}{dx}(co\sec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, x \in R - [-1,1]$

Derivatve of implicit function:

Two variables x and y occurs together in an equation f (x,y)=0,in which the dependent variable y can not be expressed in terms of x is known as implicit function.

Ex: $x^2 + y^2 = 9, e^{\tan(x+y)} = \log(x^2 - y^2)\, etc.$

Derivative of some standard function:

$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$

$or \frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$

$* \frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$

$* \frac{d}{dx}(\log y) = \frac{1}{y}\frac{dy}{dx}$

$* \frac{d}{dx}\left(e^{y^2}\right) = e^{y^2}2y\frac{dy}{dx}$

Derivative by using logarithm:

If a function in the form of $\left(f(x)\right)^{g(x)}$ then we will use log in both sides.

$Ex: x^x, \sin x^{\cos x}.\log x^{\tan x}\ etc$

$\underline{Method\ to\ find\ \frac{dy}{dx}}$

$\underline{METHOD-1}$

$let\ y = \left(f(x)\right)^{g(x)}$

$\Rightarrow \ln y = \ln\left(f(x)\right)^{g(x)} = g(x)\ln f(x)$

$\Rightarrow \frac{d(\ln y)}{dx} = \frac{d}{dx}\left(g(x)\ln f(x)\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x)\frac{d}{dx}\ln f(x) + \ln f(x)\frac{d}{dx}g(x)\ \left(u\sin g\ \frac{d}{dx}(U.V)\right)$

$\underline{METHOD-2}$

$let\ y = \left(f(x)\right)^{g(x)} = e^{g(x)\ln\left(f(x)\right)}$

$\frac{dy}{dx} = \frac{d}{dx}e^{g(x)\ln\left(f(x)\right)} = e^{g(x)\ln\left(f(x)\right)}\frac{d}{dx}\left\{g(x)\ln\left(f(x)\right)\right\} = \left(f(x)\right)^{g(x)}\left[g(x)\frac{d}{dx}\ln f(x) + \ln f(x)\frac{d}{dx}g(x)\right]$

$Derivatve\ of\ Parametric\ function:$

$If\ the\ variables\ x\ and\ y\ of\ a\ function\ y = f(x)\ can\ be\ \exp ressed\ as\ the\ function\ of\ a\ third\ variable$
$'t'\ i.e\ x = g(t)\ and\ y = h(t)\ is\ known\ as\ parametric\ function\ with\ parameter\ 't'.$

$Ex: x = at^2\ and\ y = 2at$

$\underline{Method\ for\ finding\ \frac{dy}{dx}}$

$Let\ x = g(t) \Rightarrow \frac{dx}{dt} = g^{'}(t)$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{h^{'}(t)}{g^{'}(t)}$$

Derivative of a function with respect to another function

If $y = f(x)$ and $z = g(x)$ be two differentiable function then we have to find the derivative of y with respect to z. $\left(i.e \dfrac{dy}{dz} \text{ taking } x \text{ as the parameter} \right)$

Method for finding $\dfrac{dy}{dz}$

$$y = f(x)$$
$$\Rightarrow \frac{dy}{dx} = f'(x)$$
$$z = g(x)$$
$$\Rightarrow \frac{dz}{dx} = g'(x)$$
$$\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{f'(x)}{g'(x)}$$

Assignment-1: **Find $\frac{dy}{dx}$ (Algebra of Derivative)**

1. $y = \log_x x$
2. $y = e^{3lnx}$
3. $y = 9 \times 3^x$
4. $y = x^2 + sinx + \frac{1}{x^2}$
5. $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$
6. $y = \sqrt{\frac{1-cos2x}{1+cos2x}}$
7. $y = e^{xlna} + e^{alnx} + e^{alna}$
8. $y = x^3 tanx = x^3 \frac{sinx}{cosx}$
9. $y = e^x sinx + x^n cosx$
10. $y = x^n cotx$
11. $y = x^2 sinxlnx$
12. $y = \frac{e^x}{1+sinx}$
13. $y = \frac{x+sinx}{x+cosx}$
14. $y = \frac{x+3}{x^2+1}$
15. $y = \frac{1+tanx}{1-tanx}$
16. $y = \frac{x}{1+tanx}$
17. $y = \frac{xsinx}{1+cosx}$
18. $y = \frac{x^n}{sinx}$
19. $y = \frac{1}{ax^2+bx+c}$
20. $y = xsinx + \frac{e^x}{1+x^2}$

Assignment-2: Find $\frac{dy}{dx}$ **(Differentiation of Composite Function)/ Chain Rule**

1. $y = \sin(tanx)$
2. $y = \cos(x^2 + 1)$
3. $y = \tan(secx)$
4. $y = \cot(ax + b)$
5. $y = \text{cosex}(x + \frac{\pi}{4})$
6. $y = \sin(\text{lnx})$
7. $y = e^{cotx}$
8. $y = e^{3x+5}$
9. $y = a^{lnx}$
10. $y = a^{x^2+5}$
11. $y = sin7xcos5x$
12. $y = \ln(tanx)$
13. $If\ y = \ln(sec\theta + tan\theta)\ , find \frac{dy}{d\theta}$
14. $y = \ln(\text{x}^2 + \text{a}^2)$
15. $y = log_7(log_7 x)$
16. $y = e^{xsinx}$
17. $y = 3^{xlnx}$
18. $y = sin^5 x$
19. $y = tan^3 x$
20. $y = (x^3 + 3x^2 + 7)^9$
21. $y = (x + sinx)^{11}$
22. $y = \sqrt{tanx}$
23. $y = \sqrt{ax^2 + bx + c}$
24. $y = \sqrt{a^2 - x^2}$
25. $y = \frac{1}{lnx}$
26. $y = \sqrt{\frac{1-sinx}{1+sinx}}$
27. $y = ln(sinx^2)$
28. $y = cosec(ax + b)^2$
29. $y = e^{sin\sqrt{x}}$
30. $y = \sin(\ln(sinx))$
31. $y = (lnsinx)^2$
32. $y = \sec(lnx^n)$
33. $y = lntan\left(\frac{\pi}{4} + \frac{x}{2}\right)$
34. $y = ln\left(x + \sqrt{1 + x^2}\right)$
35. $y = \frac{e^x + lnx}{sin3x}$
36. $y = e^{\sqrt{cotx}}$
37. $y = cos(lnx)^2$
38. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
39. $y = \sqrt{a^{\sqrt{x}}}$
40. $y = sin^2 xcos^2 x$

Assignment-3: **Find $\frac{dy}{dx}$ (Differentiation of Inverse Trigonometric Function)**

1. $y = sin^{-1}3x$

2. $y = sin^{-1}(cosx)$

3. $y = cos^{-1}(3x + 1)$

4. $y = tan^{-1}\sqrt{x}$

5. $y = cot^{-1}lnx$

6. $y = sec^{-1}(x + 2)$

7. $y = cosec^{-1}(x + \frac{\pi}{4})$

8. $y = \sin(msin^{-1}x)$

9. $y = a^{(sin^{-1}x)^2}$

10. $y = e^{cos^{-1}\sqrt{1-x^2}}$

11. $y = e^{tan^{-1}\sqrt{x}}$

12. $y = lntan^{-1}x$

13. $y = \sqrt{tan^{-1}\frac{x}{2}}$

14. $y = \frac{xsin^{-1}x}{\sqrt{1-x^2}}$

15. $y = (sin^{-1}x^4)^4$

16. $y = x\ sin^{-1}x + \sqrt{1 - x^2}$

17. $y = x^2cosec^{-1}\frac{1}{x}$

Assignment-4: **Find $\frac{dy}{dx}$ (Differentiation by using Substitution Method)**

1. $y = cos^{-1}\left(2x\sqrt{1-x^2}\right)$

2. $y = cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

3. $y = tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

4. $y = cos^{-1}\left(\frac{x}{\sqrt{a^2+x^2}}\right)$

5. $y = sin^{-1}\left(\frac{sinx+cosx}{\sqrt{2}}\right), \frac{-3\pi}{4} < x < \frac{\pi}{4}$

6. $y = tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

7. $y = tan^{-1}\left(\frac{4x}{1-4x^2}\right)$

8. $y = tan^{-1}\left(\frac{2a^x}{1-a^{2x}}\right)$

9. $y = cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

10. $y = tan^{-1}\left(\frac{a+btanx}{b-atanx}\right)$

11. $y = tan^{-1}\left(\frac{x-a}{x+a}\right)$

12. $y = tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

13. $y = sin\left\{2tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$

14. $y = tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

15. $y = tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

16. $y = cot^{-1}\left(\frac{1-x}{1+x}\right)$

17. $y = sin^{-1}\frac{1-x^2}{1+x^2} + sec^{-1}\frac{1+x^2}{1-x^2}$

18. $y = tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$

19. $y = tan^{-1}\left(\frac{acosx-bsinx}{bcosx+asinx}\right)$

20. $y = tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

21. $y = tan^{-1}\left(x+\sqrt{1+x^2}\right)$

22. $y = tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

23. $y = tan^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right)$

24. $y = tan^{-1}\left(\frac{cosx}{1+sinx}\right)$

25. $y = tan^{-1}\left(\sqrt{\frac{1+sinx}{1-sinx}}\right)$

26. $y = tan^{-1}(secx + tanx)$

Assignment-5: (Differentiation By Using Logarithm)

1.$y = sinx^{lnx}$

2. $y = lnx^{sinx}$

3. $y = sinx^{cos^{-1}x}$

4. $y = sinx^{tanx} + cosx^{sinx}$

5. $y = (lnx)^x + x^{lnx}$

6. $y = cos(x^x)$

7. $y = ln(x^x + cosecx)$

8. $y = lnx^{lnx}$

9. $y = (sin^{-1}x)^x$

10. $y = (tanx)^{\frac{1}{x}}$

11. $y = tanx^{cotx} + cotx^{tanx}$

12. $y = x^x + (sinx)^x$

13. $y = x^n + n^x + x^x + n^n$

14.$y = sin(x^x)$

15.$y = \frac{(x^2-1)^3(2x-1)}{\sqrt{x-3}(4x-1)}$

16.$y = \frac{\sqrt{1-x^2}(2x-3)^{\frac{1}{2}}}{(x^2+2)^{\frac{2}{3}}}$

17.$y = x^{x^x}$

18.$y = (x^x)^x$

19. $y = lnx^x + x^{lnx}$

20. $y = sinx^{tanx}$

Assignment-6: **Find $\frac{dy}{dx}$ (Differentiation of Implicit function)**

1.$x^3 + 3y^3 = 9$

2.$ax^{\frac{3}{2}} + by^{\frac{3}{2}} = c^{\frac{3}{2}}$

3.$\sqrt{x} + \sqrt{y} = \sqrt{a}$

4.$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5.$x^2 + 2xy + y^3 = 42$

6.$y^3 - 3xy^2 = x^3 + 3x^2y$

7.$tan^{-1}(x^2+y^2)=a$ 8.$e^{x-y}=ln\frac{x}{y}$ 9.$\ln(x^2+y^2)=2tan^{-1}\left(\frac{y}{x}\right)$

10. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ $then\ prove\ that\ \frac{dy}{dx}=\frac{-1}{(x+1)^2}$

11. $if\ siny=xsin(a+y)then\ prove\ that\ \frac{dy}{dx}=\frac{sin^2(a+y)}{sina}$

12.$If\ \sqrt{1-x^6}+\sqrt{1-y^6}=K(x^3-y^3)then\ prove\ that\ \frac{dy}{dx}=\frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$

13.$If\ \sqrt{1-x^2}+\sqrt{1-y^2}=K(x-y)then\ prove\ that\ \frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

14.$Find\ \frac{dy}{dx}\ if\ xyln(x+y)=1$

15.$If\ y=xsiny\ then\ prove\ that\ \frac{dy}{dx}=\frac{y}{x(1-xcosy)}$

16.$\tan(x+y)+\tan(x-y)=1, find\frac{dy}{dx}$ 17.$Find\ \frac{dy}{dx}\ if\ e^x+e^y=e^{x+y}$

18.$x^y=y^{sinx}$ 19. $x^y=e^{x-y}$ 20. $cosx^y=siny^x$

21. $x^my^n=(x+y)^{m+n}$ 22.$x^y+y^x=2$ 23. $x^yy^x=1$

24. $x^y+y^x=(x+y)^{x+y}$

25.$If\ y=xsin(a+y)then\ prove\ that\ \frac{dy}{dx}=\frac{sin^2(a+y)}{\sin(a+y)-ycos(a+y)}$

Assignment-7: **Find $\frac{dy}{dx}$ (Differentiation of Parametric Functions)**

1. $x = a\tan\theta$ and $y = a\sec\theta$

2. $x = a\sec^3\theta$ and $y = a\tan^3\theta$

3. $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

4. $x = a\sin^2\theta$ and $y = a\cos^2\theta$

5. $x = a\,(t + 1/t)$ and $y = a(t - 1/t)$

6. $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

7. $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

8. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

9. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

10. $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

11. $x = a\left\{\cos t + \frac{1}{2}\ln\tan^2\frac{t}{2}\right\}$ and $y = a\sin t$

12. $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\frac{1}{\sqrt{1+t^2}}$

13. $x = \tan^{-1}\frac{2t}{1-t^2}$ and $y = \sin^{-1}\frac{1}{1+t^2}$

14. $x = \tan^{-1}\frac{2t}{1-t^2}$ and $y = \cos^{-1}\frac{1-t^2}{1+t^2}$

Assignment-8: **(Differentiation of a Function with respect to a function)**

1.$Differentiate\ x^2\ w.r\ to\ \sqrt{x}$

2. $Differentiate\ \ln(1+x^2)\ w.r\ to\ tan^{-1}x$

3. $Differentiate\ cosx\ w.r\ to\ sinx$

4. $Differentiate\ \ sin^{-1}x\ \ w.r\ to\ cos^{-1}x$

5. $Differentiate\ tan^{-1}x\ w.r\ to\ tan^{-1}\sqrt{1+x^2}$

6. $Differentiate\ \frac{1-sinx}{1+sinx}\ w.r\ to\ \frac{1-cosx}{1+cosx}$

7. $Differentiate\ \ tan^{-1}\frac{2x}{1-x^2}\ w.r\ to\ cos^{-1}\frac{1-x^2}{1+x^2}$

8. $Differentiate\ \ sin^{-1}\frac{2x}{1+x^2}\ w.r\ to\ cos^{-1}\frac{1-x^2}{1+x^2}$

9. $Differentiate\ \ sin^{-1}2x\sqrt{1-x^2}\ w.r\ to\ sec^{-1}\frac{1}{\sqrt{1-x^2}}$

10. $Differentiate\ \ lnsinx\ w.r\ to\ \sqrt{cosx}$

11. $Differentiate\ \ tan^{-1}\frac{\sqrt{1+x^2}-1}{x}\ w.r\ to\ tan^{-1}x$

12. $Differentiate\ \ x^x\ w.r\ to\ xlnx$

13. $Differentiate\ \ x^{sin^{-1}x}\ w.r\ to\ sin^{-1}x$

14. $Differentiate\ \ tan^{-1}\frac{\sqrt{1-x^2}}{x}\ w.r\ to\ cos^{-1}2x\sqrt{1-x^2}$

15. $Differentiate\ \ tan^{-1}\frac{cosx}{1+sinx}\ w.r\ to\ sec^{-1}x$

16. $Differentiate\ \ sin^{-1}\sqrt{1-x^2}\ w.r\ to\ cot^{-1}\frac{x}{\sqrt{1-x^2}}$

HIGHER ORDER DERIVATIVE

$y = f(x) ------(1)$

Differentiate equation (1) *w.r.t* x, *we get*

$y_1 = y' = \frac{dy}{dx} = f'(x)$, *which is called* 1*st order derivative.*

Again differentiate $y_1 = y' = \frac{dy}{dx} = f'(x)$ *w.r.t* x, *we get*

$y_2 = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$, *which is called* 2*nd order derivative.*

similarly $y_3 = y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x)$

$y_4 = y'''' = \frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = f''''(x)$

$-----------------$

Let us consider a function $y_n = y^n = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^n(x)$, *which is called nth order derivative* ·

Note: 1. $y_1 = \frac{dy}{dx}$, 2. $y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

Ex: *Find* y_1 *and* y_2 *if* $y = x^9$

Ans: *given* $y = x^9$

$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(x^9\right) = 9x^8$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(9x^8\right) = 9 \times 8x^7 = 72x^7$

Ex: *Find* y_1 *and* y_2 *if* $y = \tan x$

Ans: $y = \tan x$

$y_1 = \frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sec^2 x\right) = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \times \sec x \tan x = 2\sec^2 x \tan x$

$Ex: Find\ y_1\ and\ y_2\ if\ y=\sqrt{x}$

$Ans: given\ y=\sqrt{x}$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right)=\frac{1}{2}\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}=-\frac{1}{4}x^{-\frac{3}{2}}$$

$Ex:: Find\ y_1\ and\ y_2\ if\ y=\cos 2x$

$Ans: given\ y=\cos 2x$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}\left(\cos 2x\right)=-\sin 2x\times\frac{d}{dx}\left(2x\right)=-\sin 2x\times 2=-2\sin 2x$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-2\sin 2x\right)=-2\cos 2x\times 2=-4\cos 2x$$

$Ex: Find\ y_1\ and\ y_2\ if\ y=\ln\left(3x-4\right)$

$Ans: given\ y=\ln\left(3x-4\right)$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}\left(\ln\left(3x-4\right)\right)=\frac{1}{3x-4}\frac{d}{dx}\left(3x-4\right)=\frac{1}{3x-4}\times 3=\frac{3}{3x-4}$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{3}{3x-4}\right)=3\frac{d}{dx}\left(\frac{1}{3x-4}\right)=3\left(-\frac{1}{\left(3x-4\right)^2}\right)\times 3=\frac{-9}{\left(3x-4\right)^2}$$

$Ex: Find\ y_1\ and\ y_2\ if\ x=at^2\ and\ y=2at$

$Ans: x=at^2$

$$\frac{dx}{dt}=\frac{d}{dt}\left(at^2\right)=a2t=2at$$

$y=2at$

$$\frac{dy}{dt}=\frac{d}{dt}\left(2at\right)=2a$$

$$y_1=\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{2a}{2at}=\frac{1}{t}$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{t}\right)=-\frac{1}{t^2}\frac{dt}{dx}=-\frac{1}{t^2}\times\frac{1}{2at}=-\frac{1}{2at^3}$$

$Ex: Find\ y_1\ and\ y_2\ if\ x = a\cos\theta\ ,\ y = a\sin\theta$

$Ans: x = a\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$

$y = a\sin\theta \Rightarrow \dfrac{dy}{d\theta} = \dfrac{d}{d\theta}(a\sin\theta) = a\cos\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\cot\theta) = \frac{d}{d\theta}(-\cot\theta)\frac{d\theta}{dx} = -\left(-\operatorname{cosec}^2\theta\right)\frac{1}{-a\sin\theta}\left(\because \frac{dx}{d\theta} = -a\sin\theta\right)$$

$$= -\frac{\operatorname{cosec}^3\theta}{a}$$

$Ex: If\ \ y = \tan^{-1} x\ then\ prove\ that\ \left(1+x^2\right)y_2 + 2xy_1 = 0$

$or\ \ \left(1+x^2\right)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$

$Ans: Given\ y = \tan^{-1} x$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$\Rightarrow \left(1+x^2\right)y_1 = 1\ Differentiate\ w.r.t\ x\ both\ side$

$$\Rightarrow \frac{d}{dx}\left(\left(1+x^2\right)y_1\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \left(1+x^2\right)\frac{d}{dx}(y_1) + y_1\frac{d}{dx}\left(1+x^2\right) = 0$$

$$\Rightarrow \left(1+x^2\right)y_2 + y_1 2x = 0$$

$$So\left(1+x^2\right)y_2 + 2xy_1 = 0\ \ or\ \ \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0\ \left(\because only\ replace\ y_2 = \frac{d^2y}{dx^2}\ and\ y_1 = \frac{dy}{dx}\right)$$

$Ex: If\ y = e^{m\cos^{-1}x}\ then\ prove\ that\ \left(1-x^2\right)y_2 - xy_1 - m^2 y = 0$

$or\ \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0$

$Ans: Given\ y = e^{m\cos^{-1}x}$

$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(e^{m\cos^{-1}x}\right) = e^{m\cos^{-1}x}\frac{d}{dx}\left(m\cos^{-1}x\right) = e^{m\cos^{-1}x}\, m\frac{-1}{\sqrt{1-x^2}}$

$Squaring\ both\ sides$

$y_1^{\,2} = \left(e^{m\cos^{-1}x}\right)^2 m^2\frac{1}{1-x^2} = y^2 m^2\frac{1}{1-x^2}\qquad \left(\because y = e^{m\cos^{-1}x}\right)$

$\left(1-x^2\right)y_1^{\,2} = m^2 y^2$

$\Rightarrow \frac{d}{dx}\left(\left(1-x^2\right)y_1^{\,2}\right) = \frac{d}{dx}(m^2y^2)$

$\Rightarrow \left(1-x^2\right)\frac{d}{dx}(y_1^{\,2}) + y_1^{\,2}\frac{d}{dx}\left(1-x^2\right) = m^2\frac{d}{dx}(y^2)$

$\Rightarrow \left(1-x^2\right)2y_1\frac{d}{dx}\left(y_1\right) + y_1^{\,2}\left(-2x\right) = m^2 2y\frac{dy}{dx}\qquad \left(\because \frac{d}{dx}(y^2) = 2y\frac{dy}{dx}, \frac{d}{dx}(y_1^{\,2}) = 2y_1\frac{d}{dx}\left(y_1\right) = 2y_1y_2\right)$

$\Rightarrow \left(1-x^2\right)y_1y_2 - xy_1^{\,2} = m^2 yy_1$

$\Rightarrow \left(1-x^2\right)y_2 - xy_1 = m^2 y\qquad \left(cancel\ 2\ and\ y_1\ in\ both\ sides\right)$

$So\ \left(1-x^2\right)y_2 - xy_1 - m^2 y = 0$

$Ex: If\ y = \sin\left(m\sin^{-1}x\right)\ then\ prove\ that\ \left(1-x^2\right)y_2 - xy_1 + m^2 y = 0$

$or\ \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$

$Ans: Given\ y = \sin\left(m\sin^{-1}x\right)$

$y_1 = \frac{dy}{dx} = \frac{d}{dx}\sin\left(m\sin^{-1}x\right) = \cos\left(m\sin^{-1}x\right)\frac{d}{dx}\left(m\sin^{-1}x\right) = \cos\left(m\sin^{-1}x\right)m\frac{1}{\sqrt{1-x^2}}$

$Squaring\ both\ sides$

$y_1^{\,2} = \left(\cos\left(m\sin^{-1}x\right)\right)^2 m^2\frac{1}{1-x^2} = \left(1-\left(\sin\left(m\sin^{-1}x\right)\right)^2\right)m^2\frac{1}{1-x^2} = \left(1-y^2\right)m^2\frac{1}{1-x^2}$

$\left(1-x^2\right)y_1^{\,2} = \left(1-y^2\right)m^2$

$\Rightarrow \left(1-x^2\right)y_1^{\,2} = m^2 - m^2y^2$

$\Rightarrow \frac{d}{dx}\left(\left(1-x^2\right)y_1^{\,2}\right) = \frac{d}{dx}\left(m^2\right) - \frac{d}{dx}(m^2y^2)$

$\Rightarrow \left(1-x^2\right)\frac{d}{dx}(y_1^{\,2}) + y_1^{\,2}\frac{d}{dx}\left(1-x^2\right) = -m^2\frac{d}{dx}(y^2)\qquad \left(\because \frac{d}{dx}\left(m^2\right) = 0\right)$

$\Rightarrow \left(1-x^2\right)2y_1\dfrac{d}{dx}\left(y_1\right)+y_1^{\,2}\left(-2x\right)=-m^2 2y\dfrac{dy}{dx} \quad \left(\because \dfrac{d}{dx}(y^2)=2y\dfrac{dy}{dx}, \dfrac{d}{dx}(y_1^{\,2})=2y_1\dfrac{d}{dx}\left(y_1\right)=2y_1y_2\right)$

$\Rightarrow \left(1-x^2\right)y_1y_2-xy_1^{\,2}=-m^2yy_1$

$\Rightarrow \left(1-x^2\right)y_2-xy_1=-m^2y \quad (\because cancel\ 2\ and\ y_1\ in\ both\ sides)$

$So\ \left(1-x^2\right)y_2-xy_1+m^2y=0$

$Ex: If\ y=\sin\left(p\sin^{-1}x\right)\ then\ prove\ that\left(1-x^2\right)y_2-xy_1+p^2y=0$

$or\ \left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+p^2y=0$

$Ex: If\ x=\sin t\ and\ y=\sin pt\ then\ prove\ that\left(1-x^2\right)y_2-xy_1+p^2y=0$

$or\ \left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+p^2y=0$

$Ans: Given\ x=\sin t\ and\ y=\sin pt$

$Now\ \ y=\sin pt=\sin(p\sin^{-1}x)$

Find the solution by using above example.

$Ex: Ex: If\ x=\sin t\ and\ y=\sin 2t\ then\ prove\ that\left(1-x^2\right)y_2-xy_1+4y=0$

$or\ \left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+4y=0$

$Ex: If\ y=\left(\sin^{-1}x\right)^2\ then\ show\ that\left(1-x^2\right)y_2-xy_1-2=0$

$Ans: y=\left(\sin^{-1}x\right)^2$

$y_1=\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\sin^{-1}x\right)^2=2\left(\sin^{-1}x\right)\dfrac{d}{dx}\left(\sin^{-1}x\right)=2\sin^{-1}x\dfrac{1}{\sqrt{1-x^2}}$

$Squaring\ both\ sides$

$y_1^{\,2}=\left(2\sin^{-1}x\right)^2\dfrac{1}{1-x^2}=4\left(\sin^{-1}x\right)^2\dfrac{1}{1-x^2}$

$\left(1-x^2\right)y_1^{\,2}=4y$

$Differentiate\ w.r.t\ x\ both\ sides$

$\dfrac{d}{dx}\left(\left(1-x^2\right)y_1^{\,2}\right)=\dfrac{d}{dx}4y$

$\Rightarrow \left(1-x^2\right)\dfrac{d}{dx}y_1^{\,2}+y_1^{\,2}\dfrac{d}{dx}\left(1-x^2\right)=4\dfrac{dy}{dx}$

$$\Rightarrow\left(1-x^2\right)2y_1\frac{d}{dx}\left(y_1\right)+{y_1}^2\left(-2x\right)=4y_1$$

$$\Rightarrow\left(1-x^2\right)y_1y_2-x{y_1}^2=2y_1$$

$$\Rightarrow\left(1-x^2\right)y_2-xy_1=2\quad\left(\because cancel\,2 and\ y_1\,in\,both\,sides\right)$$

$$So\left(1-x^2\right)y_2-xy_1-2=0$$

$$Ex: If\ \ y=e^{\tan^{-1}x}\ then\,show\,that\left(1+x^2\right)y_2+(2x-1)y_1=0$$

$$Ans: y=e^{\tan^{-1}x}$$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}e^{\tan^{-1}x}=e^{\tan^{-1}x}\frac{d}{dx}\left(\tan^{-1}x\right)=e^{\tan^{-1}x}\frac{1}{1+x^2}$$

$$\Rightarrow\left(1+x^2\right)y_1=y\qquad\left(\because y=e^{\tan^{-1}x}\right)$$

$$Differentiate\ w.r.t\ x\ both\ sides$$

$$\frac{d}{dx}\left(\left(1+x^2\right)y_1\right)=\frac{d}{dx}\left(y\right)$$

$$\Rightarrow\left(1+x^2\right)\frac{d}{dx}y_1+y_1\frac{d}{dx}\left(1+x^2\right)=\frac{dy}{dx}$$

$$\Rightarrow\left(1+x^2\right)y_2+y_1 2x=y_1$$

$$\Rightarrow\left(1+x^2\right)y_2+y_1 2x-y_1=0$$

$$\Rightarrow\left(1+x^2\right)y_2+(2x-1)y_1=0$$

$$Ex: If\ \ y=ax\sin x\ then\ prove\ that\ x^2y_2-2xy_1+\left(x^2+2\right)y=0$$

$$Ans: Given\ y=ax\sin x$$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}\left(ax\sin x\right)=a\frac{d}{dx}\left(x\sin x\right)=a\left(x\frac{d}{dx}\sin x+\sin x\frac{d}{dx}x\right)=a\left(x\cos x+\sin x\right)$$

$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}a\left(x\cos x+\sin x\right)=a\left(\frac{d}{dx}x\cos x+\frac{d}{dx}\sin x\right)$$

$$=a\left(x\frac{d}{dx}\cos x+\cos x\frac{d}{dx}x+\cos x\right)=a\left(x\left(-\sin x\right)+\cos x+\cos x\right)=a\left(-x\sin x+2\cos x\right)$$

$$Now\ \ x^2y_2-2xy_1+\left(x^2+2\right)y=x^2a\left(-x\sin x+2\cos x\right)-xa\left(x\cos x+\sin x\right)+\left(x^2+2\right)ax\sin x$$

$$=-ax^3\sin x+2ax^2\cos x-2ax^2\cos x-2ax\sin x+ax^3\sin x+2ax\sin x=0$$

$Ex: If\ \ y = A\cos nx + B\sin nx\, then\ prove\, that\ y_2 + n^2 y = 0$

$or\ \ \dfrac{d^2y}{dx^2} + n^2 y = 0$

$Ans: Given\ y = A\cos nx + B\sin nx$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(A\cos nx + B\sin nx\right) = \frac{d}{dx}\left(A\cos nx\right) + \frac{d}{dx}\left(B\sin nx\right) = A\left(-\sin nx \times n\right) + B\cos nx \times n$$

$$= -An\sin nx + Bn\cos nx$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-An\sin nx + Bn\cos nx\right) = \frac{d}{dx}\left(-An\sin nx\right) + \frac{d}{dx}\left(Bn\cos nx\right)$$

$$= -An\cos nx \times n + Bn\left(-\sin nx \times n\right) = -An^2\cos nx - Bn^2\sin nx$$

$$Now\ \ \frac{d^2y}{dx^2} + n^2 y = -An^2\cos nx - Bn^2\sin nx + n^2\left(A\cos nx + B\sin nx\right)$$

$$= -An^2\cos nx - Bn^2\sin nx + An^2\cos nx + Bn^2\sin nx = 0$$

$$Hence\ \frac{d^2y}{dx^2} + n^2 y = 0$$

$Ex: If\ \ y = \sin\left(\sin x\right) then\ prove\, that\ y_2 + \tan x y_1 + y\cos^2 x = 0$

$or\ \ \dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = 0$

$Ans: Given\ y = \sin\left(\sin x\right)$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}\sin\left(\sin x\right) = \cos\left(\sin x\right)\frac{d}{dx}\sin x = \cos\left(\sin x\right)\cos x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\cos\left(\sin x\right)\cos x = \cos\left(\sin x\right)\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\cos\left(\sin x\right)$$

$$= \cos\left(\sin x\right)\left(-\sin x\right) + \cos x\left(-\sin\left(\sin x\right)\right)\frac{d}{dx}\sin x = -\sin x\cos\left(\sin x\right) - \cos^2 x\sin\left(\sin x\right)$$

$$Now\ \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = -\sin x\cos\left(\sin x\right) - \cos^2 x\sin\left(\sin x\right) + \frac{\sin x}{\cos x}\cos\left(\sin x\right)\cos x + \sin\left(\sin x\right)\cos^2 x$$

$$= -\sin x\cos\left(\sin x\right) + \sin x\cos\left(\sin x\right) = 0$$

$Ex: If\ \ y = Ae^{x} + Be^{-x} then\ prove\, that\ \dfrac{d^2y}{dx^2} - y = 0$

$Ex: If\ \ y = e^{\cos x} then\ prove\, that\ \dfrac{d^2y}{dx^2} + \sin x\dfrac{dy}{dx} + y\cos x = 0$

$Ex: If\ \ y = x\sin x\, then\ \ find\ \ y_2\ at\ x = 0.$

$Ans: y = x\sin x$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(x\sin x\right) = x\cos x + \sin x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\cos x + \sin x\right) = \frac{d}{dx}\left(x\cos x\right) + \frac{d}{dx}\sin x$$

$$= x\left(-\sin x\right) + \cos x + \cos x = -x\sin x + 2\cos x$$

$At\ x = 0, y_2 = -x\sin x + 2\cos x = 0 + 2\cos 0 = 2$

$Ex: If\ \ x = a\cos^3\theta\ \ and\ y = a\sin^3\theta\ then\ \ find\ \frac{d^2y}{dx^2}.$

$$Ans: x = a\cos^3\theta \Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right) = a\frac{d}{d\theta}\cos^3\theta = a3(\cos\theta)^2\frac{d}{d\theta}\left(\cos\theta\right)$$

$$= 3a\cos^2\theta\left(-\sin\theta\right) = -3a\cos^2\theta\sin\theta$$

$$y = y = a\sin^3\theta \Rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta}\left(a\sin^3\theta\right) = a\frac{d}{d\theta}\sin^3\theta = a3(\sin\theta)^2\frac{d}{d\theta}\left(\sin\theta\right)$$

$$= 3a\sin^2\theta\left(\cos\theta\right) = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\tan\theta\right) = \frac{d}{d\theta}\left(-\tan\theta\right)\frac{d\theta}{dx} = -\sec^2\theta\frac{1}{-3a\cos^2\theta\sin\theta} = \frac{\sec^4\theta\cos ec\theta}{3a}$$

$Ex: If\ \ y = e^{\tan^{-1}x}\ then\ prove\ that\left(1 + x^2\right)y_2 + \left(2x - 1\right)y_1 = 0.$

$Ans: See\ the\ solution\ in\ the\ above\ example.$

$Ex: if\ y = \ln\left(\sec x\right)\ then\ \ find\ \ y_1\ and\ \ y_2.$

$Ans: y = \ln\left(\sec x\right)$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}\left(\ln\left(\sec x\right)\right) = \frac{1}{\sec x}\frac{d}{dx}\left(\sec x\right) = \frac{1}{\sec x}\sec x\tan x = \tan x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\tan x = \sec^2 x$$

$Ex: If\ \ y = A\cos nx + B\sin nx\ then\ prove\ that\ \frac{d^2y}{dx^2} + n^2y = 0$

$Ans: See\ the\ solution\ in\ the\ above\ example.$

$Ex: If\ \ y = \tan^{-1}x\ then\ show\ that\left(1 + x^2\right)y_2 + 2xy_1 = 0$

$Ans: See\ the\ solution\ in\ the\ above\ example.$

$Ex: If\ x = t + \frac{1}{t}\ and\ y = t - \frac{1}{t}\ then\ find\ \frac{d^2y}{dx^2}.$

$$Ans: x = t + \frac{1}{t} \Rightarrow \frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$$

$$y = t - \frac{1}{t} \Rightarrow \frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{t^2 + 1}{t^2 - 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{t^2 + 1}{t^2 - 1}\right) = \frac{d}{dt}\left(\frac{t^2 + 1}{t^2 - 1}\right)\frac{dt}{dx} = \frac{\left(t^2 - 1\right)\frac{d}{dt}\left(t^2 + 1\right) - \left(t^2 + 1\right)\frac{d}{dt}\left(t^2 - 1\right)}{\left(t^2 - 1\right)^2}\frac{dt}{dx}$$

$$= \frac{\left(t^2 - 1\right)2t - \left(t^2 + 1\right)2t}{\left(t^2 - 1\right)^2}\frac{dt}{dx} = \frac{2t^3 - 2t - 2t^3 - 2t}{\left(t^2 - 1\right)^2}\frac{dt}{dx} = \frac{-4t}{\left(t^2 - 1\right)^2} \times \frac{t^2}{t^2 - 1} = \frac{-4t^3}{\left(t^2 - 1\right)^3}$$

OR

$$Ans: x = t + \frac{1}{t}, y = t - \frac{1}{t}$$

$$Now\ y = \sqrt{\left(t + \frac{1}{t}\right)^2 - 4} = \sqrt{x^2 - 4}$$

$$y^2 = x^2 - 4 \Rightarrow \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2 - 4\right) \Rightarrow 2y\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{y - x\frac{x}{y}}{y^2} = \frac{y^2 - x^2}{y^3}$$

$$Ex: If\ y = \sin\left(\sin x\right) then\ prove\ that\ \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$$

$Ans: See\ the\ solution\ in\ the\ above\ example.$

$$Ex: If\ y = \sin\left(m\sin^{-1}x\right) then\ prove\ that\ \left(1 - x^2\right)y_2 - xy_1 + m^2y = 0$$

$Ans: See\ the\ solution\ in\ the\ above\ example.$

PARTIAL DERIVATIVE

So far we have discussed the derivative of function of one independent variable of the form $y = f(x)$.In this section ,we will discuss the derivative of functions of more than one independent variables, i.e. two variables or three variables.

Let $z = f(x, y)$, i.e. z is a function of two independent variables x and y

Then,we define two derivatives of z ,namely partial derivatives denoted by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Where $\frac{\partial z}{\partial x}$ is called partial derivative of z w.r.t x and $\frac{\partial z}{\partial y}$ is partial derivative of z w.r.t y

Mathematically, $\frac{\partial z}{\partial x}$ is the partial derivative z w.r t x where variable y taken as constant and

$\frac{\partial z}{\partial y}$ is the partial derivative of z w.r.t y where variable x taken as constant

Similarly, for $u = f(x, y, z)$,i.e. u is a function of three independent variables x,y and z,we define three partial derivatives denoted by $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$

Mathematically,

$\frac{\partial u}{\partial x}$ is partial derivative of u w.r.t x where variables y and z taken as constants

$\frac{\partial u}{\partial y}$ is partial derivative of u w.r.t y where variables x and z taken as constants

$\frac{\partial u}{\partial z}$ is partial derivative of u w.r.t z where variables x and y taken as constants

The above partial derivatives are called *first order partial derivatives*.

Second order partial derivatives:

Further differentiating partially to first order partial derivatives,we get second order partial derivatives.

i.e. $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$

are second order partial derivatives.

Notations:

For $z = f(x, y)$,

$\frac{\partial z}{\partial x}$ also denoted by f_x, $\frac{\partial z}{\partial y} = f_y$, $\frac{\partial^2 z}{\partial x^2} = f_{xx}$, $\frac{\partial^2 z}{\partial y^2} = f_{yy}$, $\frac{\partial^2 z}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 z}{\partial y \partial x} = f_{yx}$.

Ex-1: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$(i)\ z = x^2 + y^2$ $\quad (ii)\ z = xy$ $\quad (iii) z = \frac{x}{y}$ $\quad (iv) z = x^2 y - x^3 y^2 + 1$

$(v) z = \cos\ (xy)$ $\quad (vi) z = \log\ (x^2 - y^2)$ $\quad (vii) z = \sqrt{x^2 + y^2}$

$(viii) z = x^y$ $\quad (ix) z = \sin^{-1}\left(\frac{x}{y}\right)$ $\quad (x) z = e^{xy}$

$(xi)\ z = y \sin\ x$ $\quad (xii)\ z = \sin\ x . \cos\ y$

Solution:

$(i)\ z = x^2 + y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) = 2x + 0 = 2x \qquad (\because y \text{ is constant})$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) = 0 + 2y = 2y \qquad (\because x \text{ is constant})$$

$(ii)\ z = xy$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy) = y . \frac{\partial}{\partial x}(x) = y . 1 = y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy) = x . \frac{\partial}{\partial y}(y) = x . 1 = x$$

$(iii)\ z = \frac{x}{y}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y} . 1 = \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x . \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = x . (-\frac{1}{y^2}) = -\frac{x}{y^2}$$

$(iv)\ z = x^2y - x^3y^2 + 1$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial x}(x^3y^2) + \frac{\partial}{\partial x}(1) = y.2x - y^2.3x^2 + 0 = 2xy - 3x^2y^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y) - \frac{\partial}{\partial y}(x^3y^2) + \frac{\partial}{\partial y}(1) = x^2.1 - x^3.2y + 0 = x^2 - 2x^3y$$

$(v)\ z = \cos(xy)$

$$\frac{\partial z}{\partial x} = -\sin(xy).\frac{\partial}{\partial x}(xy) = -\sin(xy).y = -y.\sin(xy) \ \{\text{by chain rule}\}$$

$$\frac{\partial z}{\partial y} = -\sin(xy).\frac{\partial}{\partial y}(xy) = -\sin(xy).x = -x.\sin(xy)$$

$(vi)\ z = \log(x^2 - y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2}.\frac{\partial}{\partial x}(x^2 - y^2) = \frac{1}{x^2 - y^2}.(2x - 0) = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 - y^2}.\frac{\partial}{\partial y}(x^2 - y^2) = \frac{1}{x^2 - y^2}.(0 - 2y) = \frac{-2y}{x^2 - y^2}$$

$(vii)\ z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}}.\frac{\partial}{\partial x}(x^2 + y^2) = \frac{1}{2\sqrt{x^2 + y^2}}.(2x + 0) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}}.\frac{\partial}{\partial y}(x^2 + y^2) = \frac{1}{2\sqrt{x^2 + y^2}}.(0 + 2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

$(viii)\ z = x^y$

$$\frac{\partial z}{\partial x} = y.x^{y-1}$$

$$\frac{\partial z}{\partial y} = x^y.\log x$$

$(ix)\ z = \sin^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - (x/y)^2}}.\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{y}{\sqrt{y^2 - x^2}}.\frac{1}{y} = \frac{1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 - (x/y)^2}}.\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{y}{\sqrt{y^2 - x^2}}.\left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2 - x^2}}$$

$(x)\ z = e^{xy}$

$$\frac{\partial z}{\partial x} = e^{xy}.\frac{\partial}{\partial x}(xy) = e^{xy}.y = y\,e^{xy}$$

$$\frac{\partial z}{\partial y} = e^{xy}.\frac{\partial}{\partial y}(xy) = e^{xy}.x = x\,e^{xy}$$

$(xi)\ z = y.\sin\ x$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(y.\sin\ x) = y.\cos x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y.\sin\ x) = \sin\ x.1 = \sin\ x$$

$(xii)\ z = \sin\ x.\cos\ y$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\sin\ x.\cos\ y) = \cos\ y.\cos\ x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\sin\ x.\cos\ y) = \sin\ x.(-\sin\ y) = -\sin\ x.\sin\ y$$

Ex-2: Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

$(i) u = xyz \qquad (ii) u = xy + yz + zx \qquad (iii) u = x^2 + y^2 + z^2$

Solution:

$(i) u = xyz$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xyz) = yz,\ \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xyz) = zx,\ \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(xyz) = xy$$

$(ii) u = xy + yz + zx$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy + yz + zx) = y + 0 + z = y + z$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xy + yz + zx) = x + z + 0 = z + x$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(xy + yz + zx) = 0 + y + x = x + y$$

$(iii)\, u = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x + 0 + 0 = 2x$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = 0 + 2y + 0 = 2y$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 + z^2) = 0 + 0 + 2z = 2z$$

Ex-3:

If $u = x^2 y + y^2 z + z^2 x$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$

Solution:

$$u = x^2 y + y^2 z + z^2 x$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 y + y^2 z + z^2 x) = y.2x + 0 + z^2.1 = 2xy + z^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 y + y^2 z + z^2 x) = x^2.1 + z.2y + 0 = x^2 + 2yz$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2 y + y^2 z + z^2 x) = 0 + y^2.1 + x.2z = y^2 + 2zx$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$= (x + y + z)^2$$

Ex-4:

If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Solution:

Given $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) = \frac{1}{y} + 0 + z.\left(-\frac{1}{x^2}\right) = \frac{1}{y} - \frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) = x.\left(-\frac{1}{y^2}\right) + \frac{1}{z} + 0 = -\frac{x}{y^2} + \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) = 0 + y.\left(-\frac{1}{z^2}\right) + \frac{1}{x} = -\frac{y}{z^2} + \frac{1}{x}$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{1}{y} - \frac{z}{x^2}\right) + y\left(-\frac{x}{y^2} + \frac{1}{z}\right) + z\left(-\frac{y}{z^2} + \frac{1}{x}\right)$$

$$= \frac{x}{y} - \frac{z}{x} - \frac{x}{y} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x} = 0$$

Homogeneous Function and Euler's Theorem:

Homogeneous function :

A function $f(x, y)$ is said to be homogeneous in x and y of degree 'n' if $f(tx, ty) = t^n . f(x, y)$ where t is any constant

Euler's theorem :

If z is homogeneous of degree 'n' then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

Ex-5: Verify Euler's theorem in the following problems

$(i)\ z = xy \qquad (ii)\ z = \frac{x}{y} \qquad (iii)\ z = x^2y^2 + 4xy^3 - 3x^3y$

Solution:

$(i)\ z = xy = f(x, y)$

$\Rightarrow f(tx, ty) = tx.ty = t^2xy = t^2 f(x, y)$

$\Rightarrow z$ is homogeneous of degree 2

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy) = y,\ \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy) = x$$

Now by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + yx = 2xy = 2z$$

Hence Euler's theorem verified

$(ii)\ z = \dfrac{x}{y} = f(x,y)$

$\Rightarrow f(tx,ty) = \dfrac{tx}{ty} = \dfrac{x}{y} = t^0 \dfrac{x}{y} = t^0 f(x,y)$

$\Rightarrow z$ is homogeneous of degree 0

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}.1 = \frac{1}{y},\ \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x.\left(-\frac{1}{y^2}\right) = -\frac{x}{y^2}$$

Now by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x.\frac{1}{y} + y.\left(-\frac{x}{y^2}\right) = \frac{x}{y} - \frac{x}{y} = 0 = 0.z$$

Hence Euler's theorem verified

$(iii)\ z = x^2y^2 + 4xy^3 - 3x^3y$

$\Rightarrow z$ is homogeneous of degree 4

$$\frac{\partial z}{\partial x} = y^2.2x + 4y^3.1 - 3y.3x^2 = 2xy^2 + 4y^3 - 9x^2y$$

$$\frac{\partial z}{\partial y} = x^2.2y + 4x.3y^2 - 3x^3.1 = 2x^2y + 12xy^2 - 3x^3$$

Now by Euler's theorem,

$$\begin{aligned} x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x(2xy^2 + 4y^3 - 9x^2y) + y(2x^2y + 12xy^2 - 3x^3) \\ &= 2x^2y^2 + 4xy^3 - 9x^3y + 2x^2y^2 + 12xy^3 - 3x^3y \\ &= 4x^2y^2 + 16xy^3 - 12x^3y \\ &= 4(x^2y^2 + 4xy^3 - 3x^3y) = 4z \end{aligned}$$

Hence Euler's theorem verified

Ex - 6: Prove that if $z = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$

Sol^n : Given, $z = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$

$$\Rightarrow \sin z = \frac{x^2 + y^2}{x + y}$$

Let $u = \sin z$

Clearly, u is homogeneous of degree 1

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u \text{ ....(1)}$$

$$\text{Now, } \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin z) = \cos z.\frac{\partial z}{\partial x}, \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sin z) = \cos z.\frac{\partial z}{\partial y}$$

Hence, eq^n (1) reduces to

$$x\cos z.\frac{\partial z}{\partial x} + y\cos z.\frac{\partial z}{\partial y} = u$$

$$\Rightarrow \cos z\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = \sin z$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$$

CHAPTER-4

DIFFERENTIAL EQUATION

Differential Equation: An equation involves an independent variable, a dependent variable and the derivative of the dependent variable is known as differential equation.

Ex:$\frac{dy}{dx} + 5y = sinx$

$\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 3y = cos2x$

$ydx - xdy = 0$

$\frac{dy}{dx} = 0$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ General Solution:

Ordinary differential equation (ODE): If a differential equation involves one independent variable then it is called ordinary differential equation.

$\frac{dy}{dx} + 5y = sinx$,

$\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 3y = cos2x$

Partial differential equation(PDE): If a differential equation involves more than one independent variable then it is called partial differential equation.

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

Order of a differential equation: The highest order derivative occurring in a differential equation is known as order of a differential equation.

Degree of a differential equation: The power of the highest order derivative occurring in a differential equation after it is made free from any radicals or fractions is known as degree of a differential equation.

Ex:Find the order and degree of the following differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^3 + 4\left(\frac{dy}{dx}\right)^5 - 7\frac{dy}{dx} + 9 = 0$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}}=\frac{d^2y}{dx^2}$$

$$x^2\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^5+y^4=0$$

$$a\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}}$$

$$\frac{d^2y}{dt^2}=\frac{5y+\frac{dy}{dt}}{\sqrt{\frac{d^2y}{dt^2}}}$$

Solution of a differential equation: An equation in the form of $y=f(x)$, which satisfies to the given differential equation is known as solution of D.E.

Note: Solution of a D.E will be obtained by taking integration.

General Solution: If the solution of a D.E of nth order contains n arbitrary constants then it is called general solution.

Particular solution: A solution obtained after putting the particular values to the arbitrary constants in the general solution is known as particular solution.

Formation of a differential equation: If an equation of a curve contains **n** arbitrary constants then differentiate the given equation n times to obtain **n** number of equations, Using all these equations eliminate the constants and we will get a differential equation of **nth** order.

Ex:Form a differential equation of the following equations.

(a)$y=Ae^x+Be^{-x}$ (b)$y=ax^2+bx$ (c)$ax^2+by^2=1$ (d)$y=asint+be^t$

(e)$y=ae^{2x}+be^{-2x}$(f)$y=asinx+bcosx$ (g)$y=a\ secx$ $(g)x^2+y^2=a^2$

Techniques For Solving a D.E:

Case-1: $\frac{dy}{dx} = f(x)$

Method: $\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)dx$

Integrating both sides $\int dy = \int f(x)dx$

Case-2: $\frac{dy}{dx} = f(y)$

Method: : $\frac{dy}{dx} = f(y) \Rightarrow \frac{dy}{f(y)} = dx$

Integrating both sides $\int \frac{dy}{f(y)} = \int dx$

Case-3: (Variable separable)

$\frac{dy}{dx} = \frac{f(x)}{g(y)}$

Method: $\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow g(y)dy = f(x)dx$

Integrating both sides $\int g(y)dy = \int f(x)dx$

Case-4: (Higher order)

$\frac{d^2y}{dx^2} = f(x)$

Method: $\frac{d^2y}{dx^2} = f(x)$

Let $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$\frac{dp}{dx} = f(x) \Rightarrow dp = f(x)dx$

Integrating Both sides

$\int dp = \int f(x)dx \Rightarrow p = F(x) + c_1$

$\Rightarrow \frac{dy}{dx} = F(x) + c_1 \Rightarrow dy = \{F(x)dx + c_1dx\} Integrating\ both\ sides$

$\int dy = \int F(x)dx + \int c_1dx \Rightarrow y = h(x) + c_1x + c_2$

Or

$\frac{d^2y}{dx^2} = f(x)$

Integrate direct $\frac{dy}{dx} = F(x) + c_1$

Again integrate $y = h(x) + c_1x + c_2$

Ex: $\frac{dy}{dx} = cosx$

Ex:$\frac{dy}{dx} = 3x^2 - sinx + 7$

Ex:$\frac{dy}{dx} = \frac{1}{1+x^2}$

Ex: Solve .$\frac{dy}{dx} = 2xe^{x^2}$

Ex: Solve $\frac{dy}{dx} = cos2x\ sin3x$

Ex: Solve $\frac{dy}{dt} = \frac{tan^{-1}t\ e^{tan^{-1}t}}{1+t^2}$

Ex: Solve $\frac{dy}{du} = \frac{1}{u^2-5u+6}$

Ex: Solve $\frac{dy}{dx} = y^2 + 5y$

Ex: Solve $\frac{dy}{dx} = cotx\ tany$

Ex: Solve $sinx\frac{dy}{dx} + ycosx = 0$

Ex: Solve $x(1-y^2)dx - y(1-x^2)dy = 0$

Ex: Solve $ydy + e^{-y}xsinxdx = 0$

Ex: Solve $\frac{dy}{dx} = \sec(x+y)$

Ex:Solve $\frac{d^2y}{dx^2} = \frac{1}{2}cos^2x$

Ex: Solve $(x^2 + 7x + 12)dy + (y^2 - 6y + 5)dx = 0$

Ex: Solve $e^{-x}\frac{d^2y}{dx^2} = x$ Ex: solve $\frac{dy}{dt} = e^{3t+4y}$

Linear Differential Equation: A Differential equation is said to be linear if the dependent variable and all its derivatives occurs 1st degree only but both are not multiplied together.

Ex:$\frac{dy}{dx} + 5y = e^x$,$\frac{d^2y}{dx^2} + 7xy = x^2$

Non-linear:

$\left(\frac{dy}{dx}\right)^2 + 5y = e^x$

$y\frac{dy}{dx} + 2xy = 3x$

Here two cases are arises

Case-1:If a differential equation in the form of $\frac{dy}{dx} + Py$ =Q

Where P and Q are the function of x or constants.

Method for Solving:

Find Integrating factor (I.F) = $e^{\int Pdx}$

General Solution : $y(I.F) = \int Q(I.F)dx$

Case-2:If a differential equation in the form of $\frac{dx}{dy} + Px =$Q

Where P and Q are the function of y or constants.

Method for Solving:

Find Integrating factor (I.F) = $e^{\int Pdy}$

General Solution : $x(I.F) = \int Q(I.F)dy$

1.Solve $\frac{dy}{dx} + 5y = e^{2x}$

Ex:Solve$\frac{dy}{dx} + \frac{3}{x}y = x^2$

Ex:Solve $(1 + x^2)\frac{dy}{dx} + 2xy = cosx$

Ex:Solve $sinx\frac{dy}{dx} + 3y = cosx$

Ex:solve $(x + 2y^3)\frac{dy}{dx} = y.$

Ex:Solve $(1 + x^2)\frac{dy}{dx} + 2xy - x^3 = 0$

Ex:solve $\frac{dx}{dy} + 4x = e^{3y}$

Ex:vsolve $\frac{dx}{dy} - \frac{3}{y}x = y^3$

Ex:Solve $(1 + y^2)dx = (tan^{-1}y - x)dy$

Ex:Solve $(1 + y^2) + \left(x - e^{-tan^{-1}y}\right)\frac{dy}{dx} = 0$

Ex:Solve $(x + tany)dy = sin2ydx$

Ex:.Solve $(x + y + 1)\frac{dy}{dx} = 1$

Ex:Solve $(1 + x^2)dy + 2xydx = cotx$

Ex:Solve $xlnx\ \frac{dy}{dx} + y = 2lnx$

Ex:Solve $\frac{dy}{dx} + ysecx = tanx$

Homogeneous Differential Equation: A differential equation in the form of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,(Where f(x,y) and g(x,y) are two homogeneous function of same degree) is known as homogeneous differential equation.

Ex:$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$\frac{dy}{dx} = \frac{x^3-y^3}{x^2y}$

$\frac{dy}{dx} = \frac{y-x}{x+y}$

Method for Solving: Let us consider a homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{f(x,vx)}{g(x,vx)} = F(v) \Rightarrow x\frac{dv}{dx} = F(v) - v \Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$

Integrating both sides ,After simplification we will get the solution.

Ex: solve $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Ex: solve $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Ex: solve $\frac{dy}{dx} = \frac{y-x}{x+y}$

1.Solve $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

5.Solve $xdy - ydx = \sqrt{x^2+y^2}\,dx$
6.Solve $x^2dy + y(x+y)dx = 0$

7.Solve $(x^2+y^2)dx - 2xydy = 0$

Reference Books: Elements of Mathematics Vol.- 2 (Odisha State Bureau of Text Book preparation & Production)