KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -II

PART-1

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CONTENTS

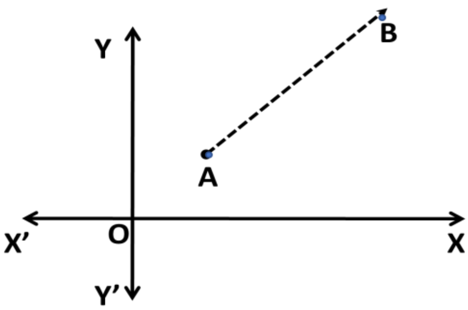
CHAPTER -1
VECTOR

Introduction:
In physics or mechanics, while studying the motion you are using a variety of quantities such as distance, displacement, speed, velocity, acceleration, mass, force, momentum, work, power, energy etc. to describe the motion. These quantities can be classified into two categories 1) Scalar quantity 2) Vector quantity.

The quantities possessing a numerical value (or magnitude) are called as scalar quantities.
Examples: Distance, speed, mass, work etc.

The quantities possessing magnitude as well as direction are called as vector quantities.
Examples: displacement, velocity, acceleration, force, momentum etc.

Representation of vectors:



In a plane the vector directed from a point A to B is denoted as $\overrightarrow{AB}$.Here the point A is called as initial point and B is the terminal point or (Final point) .

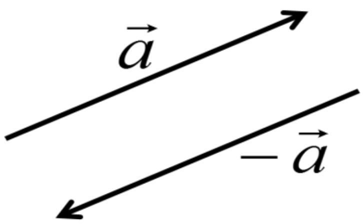
Notes: The length of a vector $\overrightarrow{AB}$ is the distance or magnitude of the vector $\overrightarrow{AB}$. It is denoted as$|\overrightarrow{AB}|$.
It is a scalar quantity , which is always positive.
Vectors are also denoted by using the lowercase bold alphabets **a, b, c** etc. or $\vec{a}, \vec{b}, \vec{c}$, etc.

Types of vectors:

Null vector:
Vector with magnitude zero (0) is called as a null vector or zero vector. It is written as $\vec{O}$.
Example: Every point is a null vector.

Unit vector: A vector with magnitude unity (1) is called as unit vector. If $\vec{a}$ be a vector, then the unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ and given by : $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Co-initial vectors: Vectors starting from a point are called as co-initial vectors.

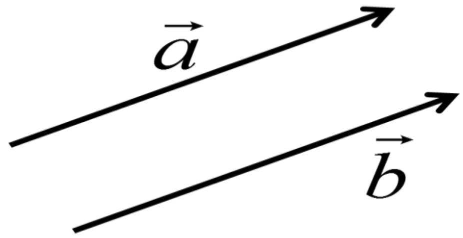


Negative of a vector: A vector having same magnitude but opposite direction is known as negative vector.

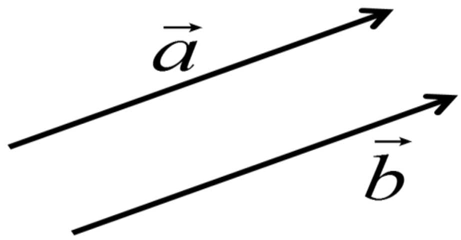
If $\vec{a}$ be a vector, then the negative of $\vec{a}$, written as -$\vec{a}$
Note: the magnitudes of $\vec{a}$ and -$\vec{a}$ are same. i.e. $|\vec{a}| = |-\vec{a}| = a$

Scalar multiplication of a vector:
Suppose k is a scalar quantity and $\vec{a}$ be a vector quantity, then the multiplication k$\vec{a}$ is a new vector called as scalar multiplication of $\vec{a}$. The vector k$\vec{a}$ is a vector whose length is $|k|$ times that of $\vec{a}$. The direction of k$\vec{a}$ will be same as of $\vec{a}$ if k is positive and opposite of $\vec{a}$ if k is negative.

Parallel vectors:
Two vectors are said to be parallel if both are of either same or opposite direction. In otherwords $\vec{a}$ and $\vec{b}$ will be parallel if $\vec{a}$ and $\vec{b}$ are scalar multiple of each other i.e. $\vec{a} = k\vec{b}$,where k is scalar.



Equal vectors:
Two vectors are said to be equal if they have same magnitude as well as same direction.



Like and Unlike vectors
Two vectors are said to be like if they are parallel but in same direction.
Two vectors are said to be unlike if they are parallel but in opposite direction.

Note: Like and Unlike vectors may be of same or different magnitudes.

Collinear of vectors:

Two vectors are said to be collinear if one is scalar multiple of other and they have a common point.
i.e $\vec{a} = k\vec{b}, where\ k\ is\ a\ scalar$

Collinear of three points:
Three points A,B and C are said to be collinear if they lie in same line and the condition is

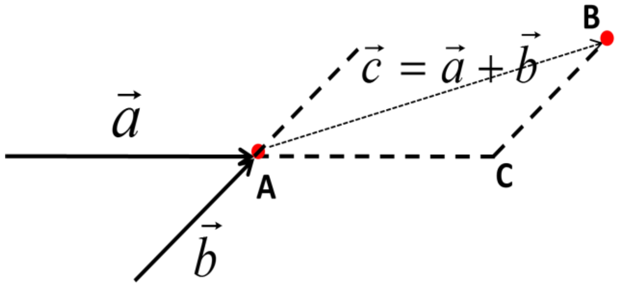
$$\overrightarrow{AB} = k\overrightarrow{AC}$$

Or

$$\overrightarrow{AB} = k\overrightarrow{BC}$$

Addition and Subtraction of vectors:

If $\overrightarrow{AC} = \vec{a}$ and $\overrightarrow{CB} = \vec{b}$ be any two vectors and join $\overrightarrow{AB}$.



Here the terminal point of $\overrightarrow{AC}$ is same as intial point of $\overrightarrow{CB}$. So $\overrightarrow{AB}$ is the sum of the vectors of $\overrightarrow{AC}$ and $\overrightarrow{CB}$.
i.e $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$=$\vec{a} + \vec{b}$, which is called triangle law of vector addition.

Subtraction of vectors:

The difference of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} - \vec{b}$ and which is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Parallelogram law of vector addition:

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ join $\overrightarrow{BC}$ and $\overrightarrow{AC}$ and make a parallelogram OACB. Join the diagonals $\overrightarrow{OC}$ and $\overrightarrow{BA}$.
Here $\overrightarrow{OA} = \vec{a} = \overrightarrow{BC}$ and $\overrightarrow{OB} = \vec{b} = \overrightarrow{AC}$
By using triangle law of vector addition we get $\overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{BA} = \vec{a} - \vec{b}$

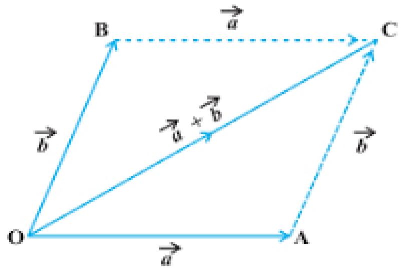


Fig. 5

Algebra of vectors:

1. Vector addition is commutative i.e. $\vec{a}$+$\vec{b}$= $\vec{b}$+$\vec{a}$
2. Vector addition is associative i.e. . $(\vec{a}$+$\vec{b}) + \vec{c}$= $\vec{a} + (\vec{b}$+$\vec{c})$
3. If m, n be any scalars, then $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$, $m(n\vec{a}) = n(m\vec{a}) = mn(\vec{a})$.
4. $1\vec{a} = \vec{a}$, $0\vec{a} = \vec{O}$

Position vector of a point in space:

If P be any point on the space and O be a fixed point (called origin) then the vector $\overrightarrow{OP} = \vec{r}$(say)
Is called the position vector of the point P with respect to origin O.
Which is written as P($\vec{r}$),(read as P be a point having position vector $\vec{r}$)

Note:
If A($\vec{a}$) and B($\vec{b}$) are two position vectors then $\overrightarrow{AB} = \vec{b} - \vec{a}$

Note:
If P(x,y) be any point on the plane and O be origin then

$$\overrightarrow{\boldsymbol{OP}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath}}$$

Where $\hat{\imath}$ *and* $\hat{\jmath}$ are the unit vectors along the direction of X-axis and Y-axis.

Note:
If P(x,y,z) be any point on the space and O be origin then
$\overrightarrow{\boldsymbol{OP}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}$

where, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along X-axis, Y-axis and Z-axis respectively.

Note:
If $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$
And
$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

Vector joining two points:
The vector joining two points A(x_1, y_1, z_1) and B(x_2, y_2, z_3) can be obtained by using the triangle law of addition of vectors
i.e. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The coefficients x_2-x_1, y_2-y_1 and z_2-z_1 are called as the scalar components of the vector $\overrightarrow{AB}$ in the direction of X-axis, Y-axis and Z-axis respectively. And the vectors $(x_2 - x_1)\hat{\imath}, (y_2 - y_1)\hat{\jmath} and\ (z_2 - z_1)\hat{k}$ are called as vector components of the vector $\overrightarrow{AB}$ in the direction of X-axis, Y-axis and Z-axis respectively.

Problem: Find $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $5\vec{a}$ and $2\vec{a} - 7\vec{b}$, where $\vec{a} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$.

Example: Find the position vector of the point (2, 3, -5).
Ans: $2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$.

Example: Find the vector from the point (2, -3, 4) to (6, 4, 1) and hence find the modulus of this vector.

Example: Find the value of m if the modulus of the vector joining A(0, 1, -2) and B(-2, 3, m) is $\sqrt{8}$.

Example: Find the unit vector in the direction of the vector $2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$.

Problem: Find the unit vector in the direction of the sum of the vectors $\vec{a}\ and\ \vec{b}$ where, $\vec{a} = -3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ and $\vec{b} = \hat{\imath} + 5\hat{\jmath} + 3\hat{k}$.
Example: Find the value of 'm' if the vectors $-3\hat{\imath} + m\hat{\jmath} - 8\hat{k}$ and $15\hat{\imath} + 2\hat{\jmath} + 40\hat{k}$ are parallel.
Example: Prove that the points (2, 1, -1), (3, -2, 1) and (8, -17, 11) are collinear.
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Questions carrying 2 marks

1. Find the position vector of the point A (2, -3). **Ans**: $2\hat{i} - 3\hat{j}$
2. Find the vector joining points A (1, -3) and B (-5, 4). **Ans**: $-6\hat{i} + 7\hat{j}$
3. Find the length of the vector joining P (2, -1) and Q (5,-4). **Ans** : $3\sqrt{2}$
4. Find the unit vector in the direction of the vector $\hat{i} + \hat{j} + \hat{k}$. **Ans** : $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$
5. For what value of 'a' the vectors $2\hat{i} - 3\hat{j}$ and $a\hat{i} - 6\hat{j}$ are parallel. **Ans** : 4

6. If the vectors $\vec{a} = \alpha\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ are parallel, find the value *of* α. **Ans**: α=-3
7. Find a unit vector parallel to the sum of vectors $3\hat{i} - 2\hat{j} + \hat{k}$ *and* $-2\hat{i} + \hat{j} - 3\hat{k}$ **Ans**: $\frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$

<u>Question carrying 5 marks</u>

1. Prove by vector method that the points A(2,6,3), B(1,2,7) and C(3,10,-1) are collinear.
2. Prove that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \vec{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ form the sides of a right angled triangle.

Product of vectors:

The product of vectors can be defined using two special symbols i.e. (•) and (X).

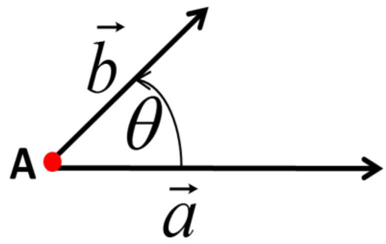
a. Scalar Product or Dot product

b. Vector Product or Cross Product

Scalar product (Dot product)

Definition:

Let $\vec{a}$ and $\vec{b}$ be any two coinitial vectors and θbe the angle between them measured from $\vec{a}$ to $\vec{b}$. Then the scalar product (or dot product) of $\vec{a}$ and $\vec{b}$ written as $\vec{a}\cdot\vec{b}$ and defined by



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta = abcos\theta$$

Similarly

Angle between two vectors

Formula:

Angle between the vectors $\vec{a}$ and $\vec{b}$ is given by:

$$cos\theta = \pm\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \theta = cos^{-1}\left(\pm\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Condition-1:If two vectors are parallel then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| = ab$

Condition-2: If two vectors are perpendicular then $\vec{a}\cdot\vec{b} = 0 = \vec{b}.\vec{a}$

Properties of Dot product

a. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

b. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

c. If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}\cdot\vec{b} = 0$.

d. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Distributive over addition)

Notes:

1. $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$ (Since $\hat{\imath},\hat{\jmath},\hat{k}$ are perpendicular)

2. $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$

3. If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Geometrical meaning of dot product:(Scalar and vector projection)

Scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Scalar projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

vector projection of $\vec{a}$ on $\vec{b} = \frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

vector projection of $\vec{b}$ on $\vec{a} = \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}$

Work done:

$$W = \vec{F}\cdot\vec{S}, where\ \vec{S} = \overrightarrow{AB}\ (\vec{F} = force)$$

Example: Find the scalar product of the vectors $\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $-2\hat{\imath} + \hat{\jmath} - 3\hat{k}$
Example: Find the value of δ if the vectors $\hat{\imath} - \hat{\jmath} - 2\hat{k}$ and $\hat{\imath} + \delta\hat{\jmath} - 3\hat{k}$ are perpendicular to each other.
Example: Find the scalar projection and vector projection of $\vec{b}\ on\ \vec{a}$ where $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 4\hat{k}, \vec{b} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$.
Example: Find the acute angle between the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

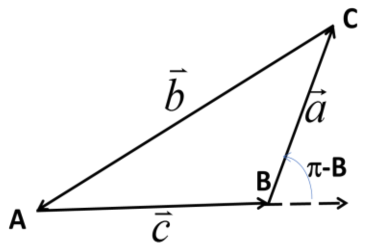
Questions carrying 2 marks

1. Find $\vec{a}.\vec{b}$ if (i) $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j}$,(ii) $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}\ and\ \vec{b} = \hat{i} - \hat{j} + \hat{k}$
2. Find the value of $'\lambda'$ if $\vec{a} = 6\hat{i} + \lambda\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$ are perpendicular to each other.
3. Show that the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - 3\hat{j} + \hat{k}$ are at right angles .

Question carrying 5 marks

1. Find the value of λ if $\vec{a} = (2,-2,1)$ and $\vec{b} = (0,2\lambda,1)$ are perpendicular.
2. Find the angle between the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}\ and\ \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$
3. Find scalar and vector projections of $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ on $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.
4. Find the work done by force $\vec{F} = 5\hat{i} - 2\hat{j} + 3\hat{k}$ which displaces a particle from A (1,-2,2) to B (3,1,5).
5. Prove by vector method that in any triangle ABC.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



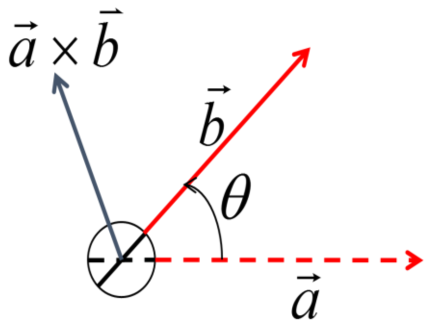
Proof: Assume the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to represent the sides of a triangle taken in order (Refer figure).

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$
$\Rightarrow \vec{a} = -(\vec{b} + \vec{c})$
$\Rightarrow \vec{a}\cdot\vec{a} = -(\vec{b} + \vec{c})\cdot\vec{a} = (\vec{b} + \vec{c})\cdot(\vec{b} + \vec{c})$
$\Rightarrow \vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{b}$

6. Prove by vector method that an angle inscribed in a semicircle is right angle.
7. If the sum of two unit vectors is a unit vector, then prove that the magnitude of their difference will be $\sqrt{3}$.
8. In a triangle ABC prove by vector method that a=bcosC + c cosB

Vector Product (Cross Product)

Definition of vector product:



Let $\vec{a}$ and $\vec{b}$ be any two coinitial vectors and θbe the angle between them measured from $\vec{a}$ to $\vec{b}$. Then the vector product (or cross product) of $\vec{a}$ and $\vec{b}$ written as $\vec{a}\times\vec{b}$ and defined by

$\vec{a}\times\vec{b} = \left(|\vec{a}||\vec{b}|sin\theta\right)\hat{n} = (absin\theta)\hat{n}$

Here $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. The direction of $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Hence $\hat{n}$ is in the direction of $\vec{a}\times\vec{b}$.

$|\vec{a}\times\vec{b}| = \left||\vec{a}||\vec{b}|sin\theta\hat{n}\right| = absin\theta\ |\hat{n}| = absin\theta$

Now $\vec{b}\times\vec{a} = \left(|\vec{a}||\vec{b}|\sin(-\theta)\right)\hat{n} = -absin\theta\ \hat{n}$

$|\vec{b}\times\vec{a}| = \left|-|\vec{b}||\vec{a}|sin\theta\hat{n}\right| = absin\theta\ |\hat{n}| = absin\theta$

So $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ but $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{a}|$

Note: $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

Sine of the angle between the vectors $\vec{a}$ and $\vec{b}$:

$sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

Condition-1: If two vectors are parallel then $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{a}| = 0\ or\ \vec{a}\times\vec{b} = \vec{0}$

Condition -2: If two vectors are perpendicular then $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|$=ab

Properties of Cross product

a. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

b. $\vec{a}\times\vec{a} = \vec{0}$

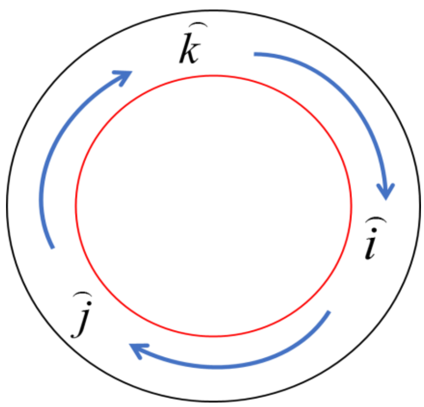
c. $\vec{a}\times\left(\vec{b}+\vec{c}\right) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ (Distributive over addition)

Notes:

1. $\hat{\imath}\times\hat{\jmath} = \hat{k}$ (Since $\hat{\imath},\hat{\jmath},\hat{k}$ are perpendicular)
2. $\hat{\jmath}\times\hat{k} = \hat{\imath}$
3. $\hat{k}\times\hat{\imath} = \hat{\jmath}$
4. $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = \vec{0}$
5. If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$,

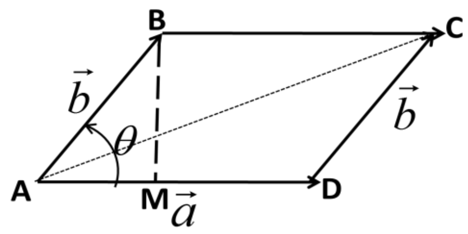
then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_2b_3 - b_2a_3)\hat{\imath} - (a_1b_3 - b_1a_3)\hat{\jmath} + (a_1b_2 - b_1a_2)\hat{k}$



Geometrical meaning of cross product:

Area of a parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$



If $\vec{a}$ and $\vec{b}$ represent the two sides of a parallelogram taken in order, then the area of the triangle will be $\frac{1}{2}|\vec{a} \times \vec{b}|$

Unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$:

The unit vector *perpendicular* to the vectors $\vec{a}$ *and* $\vec{b}$ will be $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Note:The vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$ will be $\vec{a} \times \vec{b}$

Note:Area of a triangle with vertices A, B and C is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

Example: Find the vector product of the vectors $\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $-\hat{\imath} + \hat{\jmath} - 3\hat{k}$.

Ex: Find $\vec{a} \times \vec{b}$ where $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 7\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

Example: Find the vector perpendicular to both $5\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

Example: Find the unit vector perpendicular to both $\hat{\imath} + 2\hat{\jmath}$ and $\hat{\imath} + 3\hat{\jmath} + \hat{k}$.

Example: Find the area of the parallelogram whose adjacent sides are $\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ and $\hat{\imath} - 3\hat{\jmath} + \hat{k}$.

Example: Find the area of the triangle whose vertices are at A(1, -1, 3), B(2, 1, 0) and C(3, 1, -1)

Example: Find the sine of the angle between the vectors $3\hat{\imath} - 4\hat{\jmath}$ and $6\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

<u>Question carrying 2 marks</u>

1. Find the vector perpendicular to both of the vectors $\hat{i} + \hat{j}$ *and* $\hat{j} + \hat{k}$. **Ans**: $\hat{i} - \hat{j} + \hat{k}$

2. Find the unit vector perpendicular to both the vectors $\hat{i} + \hat{j}$ and $\hat{i} - \hat{k}$.

   Ans: $\frac{-1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$

3. Find the unit vector perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 3\hat{k}$

   Ans $\frac{2}{\sqrt{110}}\hat{i} - \frac{9}{\sqrt{110}}\hat{j} - \frac{5}{\sqrt{110}}\hat{k}$

4. Find area of parallelogram whose adjacent sides are given by vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$ **Ans**: $8\sqrt{3}$

5. Find area of the triangle whose sides are given by vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

   Ans: $4\sqrt{3}$

6. Find area of the triangle whose vertices are A(1,-2,3), B(3,1,2), C(2,3,-1).

   Ans: $\frac{7\sqrt{3}}{2}$

7. Find sine angle between the vectors $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} + 2\hat{k}$

   Ans: $\frac{\sqrt{171}}{14}$

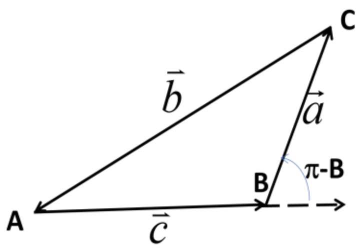
8. Find the area of the parallelogram whose diagonals are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -3\hat{i} + 4\hat{j} - \hat{k}$ **Ans**: $\frac{3\sqrt{30}}{2}$

9. Prove that $\left(\vec{a} \times \vec{b}\right)^2 = a^2b^2 - \left(\vec{a}.\vec{b}\right)^2$

10. Prove that by vector method in any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Hints: Assume the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to represent the sides of a triangle taken in order (Refer figure).



$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}=-(\vec{b}+\vec{c})$ ------------ (a)

$\Rightarrow \vec{a}\times\vec{a}=-(\vec{b}+\vec{c})\times\vec{a}=-(\vec{b}\times\vec{a})-(\vec{c}\times\vec{a})$

$\Rightarrow \vec{0}=-(\vec{b}\times\vec{a})-(\vec{c}\times\vec{a})$

$\Rightarrow (\vec{a}\times\vec{b})=(\vec{c}\times\vec{a})$ ------------ (b)

Similarly, take the cross product with $\vec{b}$ on both sides of equation (a) and get

$\Rightarrow (\vec{a}\times\vec{b})=(\vec{b}\times\vec{c})$ ------------ (c)

Comparing Equation (b) and (c) in getting

$(\vec{a}\times\vec{b})=(\vec{b}\times\vec{c})=(\vec{c}\times\vec{a})$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$

$\Rightarrow absin(\pi-C)=bcsin(\pi-A)=casin(\pi-B)$

$\Rightarrow absinC=bcsinA=casinB$

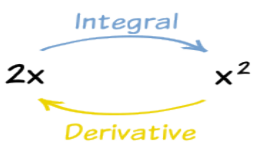
Now divide ‘abc’ on both sides to get the answer.

CHAPTER -2
INTEGRATION

Integration is an inverse process of differentiation or antiderivative process.

If the derivative of $F(x)$ is $f(x)$, then the antiderivative or integral of $f(x)$ is $F(x)$.

Let $\frac{d}{dx}\left(F(x)\right)=f(x) \Rightarrow$ integration of $f(x)=F(x)$



Again $\frac{d}{dx}\left(F(x)+c\right)=f(x) \Rightarrow \int f(x)\,dx=F(x)+c$

Where c is called constant of integration.

Here $f(x)$ is called integrand and $F(x)$ is called integral.

But dx represents integration with respect x.

The symbol $\int$ represents sign of integration.

Ex: We have $\frac{d}{dx}(\sin x)=\cos x \Rightarrow \int \cos x\,dx=\sin x+c$

FORMULAS :

$F-1 \int x^n dx=\frac{x^{n+1}}{n+1}+c \, (n \neq -1)$

Ex : Evaluate $\int x^9 dx$

Ans : $\int x^9 dx=\frac{x^{9+1}}{9+1}+c=\frac{x^{10}}{10}+c$

Ex : Evaluate $\int x^{\frac{5}{2}} dx$

Ans : $\int x^{\frac{5}{2}} dx=\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+c=\frac{2}{7}x^{\frac{7}{2}}+c$

Ex : Evaluate $\int x^{-\frac{5}{2}} dx$

Ans : $\int x^{-\frac{5}{2}} dx=\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}+c=-\frac{2}{3}x^{-\frac{3}{2}}+c$

$Ex: Evaluate \int \frac{1}{x^7} dx$

$Ans: \int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-6}}{-6} + c = -\frac{x^{-6}}{6} + c$

$Ex: Evaluate \int \frac{1}{x\sqrt{x}} dx$

$Ans: \int \frac{1}{x\sqrt{x}} dx = \int \frac{1}{x^{\frac{3}{2}}} dx = \int x^{-\frac{3}{2}} dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -2x^{-\frac{1}{2}} + c$

(IMP)

$F-2 \int e^x dx = e^x + c$

$F-3 \int a^x dx = \frac{a^x}{\ln a} + c$

$Ex: Ealuate \int 3^x dx$

$Ans: \int 3^x dx = \frac{3^x}{\ln 3} + c$

$F-4 \int \sin x \, dx = -\cos x + c$

$F-5 \int \cos x \, dx = \sin x + c$

$F-6 \int \sec^2 x \, dx = \tan x + c$

$F-7 \int \sec x \tan x \, dx = \sec x + c$

$F-9 \int \cos ec^2 x \, dx = -\cot x + c$

$F-10 \int \cos ecx \cot x \, dx = -\cos ecx + c$

$F-11 \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$F-12 \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$F-13 \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

$F-14 \int \frac{1}{x} dx = \ln|x| + c$

$Note: \int x dx = \frac{x^2}{2} + c$

$\int dx = x + c$

Algebra of integration:

F-1: $\int \left(f(x) + g(x)\right)dx = \int f(x)dx + \int g(x)dx$

Ex: $\int (sinx + cosx - e^x - sec^2x)\, dx$

$$= \int sinxdx + \int cosxdx - \int e^x dx - \int sec^2xdx$$

$= -cosx + sinx - e^x - tanx + c$

F-2: $\int kf(x)dx = k\int f(x)dx,$ $\quad$ *where k is a constant*.

Ex: $\int 5x^9dx = 5\frac{x^{10}}{10} + c = \frac{x^{10}}{2} + c$

Ex: $\int 5dx = 5x + c$

Ex: $\int \frac{7}{x}dx = 7ln|x| + c$

Ex: $\int 3sinxdx = -3cosx + c$

Ex: $\int \frac{5}{\sqrt{1-x^2}}dx = 5sin^{-1}x + c$

Ex: $\int \frac{7}{1+x^2}dx = 7tan^{-1}x + c$

Ex: $\int (3sinx - 4cosec^2x + 9sec^2x - 5e^x - 5)dx$

$$= \int 3sinxdx - \int 4cosec^2xdx + \int 9sec^2xdx - \int 5e^xdx - \int 5dx$$

$= -3cosx + 4cotx + 9tanx - 5e^x - 5x + c$

F-3: $\frac{d}{dx}\{\int f(x)dx\} = f(x)$,The differentiation of an integral is the integrand itself.

Ex : $\frac{d}{dx}\{\int sinxdx\} = sinx$

Ex: $\int \frac{1}{cos^2x}dx = \int sec^2xdx = tanx + c$

Ex: $\int \frac{1}{sin^2x}dx = \int cosec^2xdx = -cotx + c$

Ex: $\int \frac{1}{1-sin^2x}dx = \int \frac{1}{cos^2x}dx = \int sec^2xdx = tanx + c$

Ex: $\int \frac{1}{1-cos^2x}dx = \int \frac{1}{sin^2x}dx = \int cosec^2xdx = -cotx + c$

Ex:$\int \frac{cosx}{sin^2x}dx = \int cotx\ cosecxdx = -cosecx + c$

Ex:$\int \frac{sinx}{cos^2x}dx = \int tanx\ secxdx = secx + c$

IMP

Ex:$\int tan^2xdx = \int (sec^2x - 1)dx = \int sec^2xdx - \int 1\,dx = tanx - x + c$

Ex:$\int cot^2xdx = \int (cosec^2x - 1)dx = \int cosec^2xdx - \int 1\,dx = -cotx - x + c$

Ex: $\int \frac{1-sinx}{cos^2x}dx = \int \frac{1}{cos^2x}d - \int \frac{sinx}{cos^2x}dx$

$$= \int sec^2xdx - \int tanx\ secxdx = tanx - secx + c$$

Ex:$\int \frac{1-sin^3x}{sin^2x}dx = \int \frac{1}{sin^2x}dx - \int \frac{sin^3x}{sin^2x}dx = \int cosec^2x\,dx - \int sinxdx$

$= -cotx + cosx + c$

Ex:$\int \frac{cos2x}{cos^2xsin^2x}dx = \int \frac{cos^2x-sin^2x}{cos^2xsin^2x}dx = \int \frac{cos^2x}{cos^2xsin^2x}dx - \int \frac{sin^2x}{cos^2xsin^2x}dx$

=$\int cosec^2x\,dx - \int sec^2x\,dx = -cotx - tanx + c$

Ex:$\int \frac{1}{cos^2xsin^2x}dx = \int \frac{cos^2x+sin^2x}{cos^2xsin^2x}dx = \int \frac{cos^2x}{cos^2xsin^2x}dx + \int \frac{sin^2x}{cos^2xsin^2x}dx$

=$\int cosec^2x\,dx + \int sec^2x\,dx = -cotx + tanx + c$

Ex:$\int \frac{cos2x}{cosx+sinx}dx = \int \frac{cos^2x-sin^2x}{cosx+sinx}dx = \int \frac{(cosx+sinx)(cosx-sinx)}{cosx+sinx}dx$

=$\int (cosx - sinx)dx = \int cosxdx - \int sinxdx = sinx + cosx + c$

Ex:$\int (tanx + cotx)^2dx = \int (tan^2x + cot^2x + 2tanx\ cotx)\,dx$

=$\int (sec^2x - 1 + cosec^2x - 1 + 2)\,dx$=$\int sec^2xdx + \int cosec^2xdx$

=$tanx - cotx + c$

Ex:$\int \frac{x^2}{1+x^2}dx = \int \frac{x^2+1-1}{1+x^2}dx = \int \frac{x^2+1}{1+x^2}dx - \int \frac{1}{1+x^2}dx = \int 1dx - \int \frac{1}{1+x^2}dx$

= $x - tan^{-1}x$+c

Ex:$\int \frac{x^4}{1+x^2}dx = \int \frac{x^4-1+1}{1+x^2}dx$

$= \int \frac{x^4-1}{1+x^2} dx + \int \frac{1}{1+x^2} dx = \int \frac{(x^2+1)(x^2-1)}{1+x^2} dx + \int \frac{1}{1+x^2} dx$

$=\int (x^2 - 1)\, dx + \int \frac{1}{1+x^2} dx = \int x^2 dx - \int 1\, dx + \int \frac{1}{1+x^2} dx = \frac{x^3}{3} - x + tan^{-1}x$+c

Ex:$\int \frac{x^6}{1+x^2} dx = \int \frac{x^6+1-1}{1+x^2} dx = \int (\frac{(x^2)^3+1^3}{1+x^2} - \frac{1}{1+x^2}) dx$

$=\int \frac{(x^2+1)(x^4-x^2+1)}{1+x^2} dx - \int \frac{1}{1+x^2} dx = \int (x^4 - x^2 + 1)\, dx - \int \frac{1}{1+x^2} dx$

$=\int x^4 dx - \int x^2 dx + \int dx - \int \frac{1}{1+x^2} dx = \frac{x^5}{5} - \frac{x^3}{3} + x - tan^{-1}x + c$

IMP:

Ex:$\int \frac{1}{1+sinx} dx = \int \frac{1-sinx}{(1+sinx)(1-sinx)} dx = \int \frac{1-sinx}{(1-sin^2x)} dx$

$=\int \frac{1-sinx}{cos^2x} dx = \int \frac{1}{cos^2x} d - \int \frac{sinx}{cos^2x} dx$

$= \int sec^2x dx - \int tanx\ secx dx = tanx - secx + c$

Ex:$\int \frac{sinx}{1+sinx} dx = \int \frac{sinx(1-sinx)}{(1+sinx)(1-sinx)} dx = \int \frac{sinx-sin^2x}{(1-sin^2x)} dx$

$=\int \frac{sinx-sin^2x}{cos^2x} dx = \int \frac{sinx}{cos^2x} d - \int \frac{sin^2x}{cos^2x} dx = \int tanx\ secx\ dx - \int tan^2x dx$

$= \int tanx\ secx\ dx - \int (sec^2x - 1) dx = secx - tanx + x + c$

Assignment:

Ex:$\int \frac{cosx}{1-cosx}dx$

Ex:$\int \frac{cosx}{1+cosx}dx$

Ex:$\int \frac{sinx}{1-sinx}dx$

Ex:$\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

Ex:$\int \frac{x^5+5x^2-2x+7}{x^3}dx$

Ex:$\int (e^{xloga}+e^{alogx}+e^{aloga})\,dx$

Ex:$\int \frac{3-2cosx}{sin^2x}dx$

Ex:$\int \sqrt{1-sin2x}dx$

Ex:$\int \sqrt{1+sin2x}dx$

Ex:$\int \sqrt{1-cos2x}dx$

Ex:$\int \sqrt{1+cos2x}dx$

Ex:$\int \frac{cosecx}{cosecx-cotx}dx$

Ex:$\int 3^{x-2}dx$

Ex:$\int \frac{1}{1-sinx}dx$

Ex:$\int \frac{1}{1+cosx}dx$

Ex:$\int \frac{1}{1-cosx}dx$

INTEGRATION BY SUBSTITUTIONS

Integration by substitution will be used to solve the integration easily by using suitable substitution.

If the integrand in the form of $\int \boldsymbol{f(x)f'(x)dx}$

How to solve: $\int f(x)f'(x)dx$

Let f(x) = t , take derivative in both sides

$$\Rightarrow \frac{d}{dx}f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$$

So $\int f(x)f'(x)dx = \int tdt = \frac{t^2}{2} + c = \frac{(f(x))^2}{2} + c$

Ex: Evaluate $\int sinx\ cosx\ dx$

Ans: $\int sinx\ cosx\ dx$

$let\ sinx = t$

$$\Rightarrow \frac{d}{dx}sinx = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$$

So $\int sinx\ cosx\ dx = \int t\ dt = \frac{t^2}{2} + c = \frac{sin^2x}{2} + c$

$\int \boldsymbol{(f(x))^n f'(x)dx}$

How to solve:

Let f(x) = t, take derivative in both sides

$$\Rightarrow \frac{d}{dx}f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$$

$$= \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{(f(x))^{n+1}}{n+1} + c$$

Ex: $Evaluate \int sin^5x\ cosx\ dx$

Ans: $\int sin^5x\ cosx\ dx$

$let\ sinx = t$

$\Rightarrow \frac{d}{dx} sinx = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$

So $\int sin^5x\ cosx\ dx = \int t^5 dt = \frac{t^6}{6} + c = \frac{sin^6x}{6} + c$

Ex: $\int tan^3x\ sec^2x\ dx$

Ans: $\int tan^3x\ sec^2x\ dx$

$let\ tanx = t$

$\Rightarrow \frac{d}{dx} tanx = \frac{dt}{dx} \Rightarrow sec^2x\ dx = dt$

So $\int tan^3x\ sec^2x\ dx = \int t^3 dt = \frac{t^4}{4} + c = \frac{tan^4x}{4} + c$

$\int \boldsymbol{f(g(x))g'(x)dx}$

How to solve:

Let g(x) = t, take derivative in both sides

$\Rightarrow \frac{d}{dx} g(x) = \frac{dt}{dx} \Rightarrow g'(x) = \frac{dt}{dx} \Rightarrow g'(x)dx = dt$

$$so \int f(g(x))g'(x)dx$$

$$= \int f(t)dt = F(t) + c = F(g(x)) + c \quad [\because \int f(x)dx = F(x) + c$$

Ex: $Evaluate \int \cos(sinx)\ cosxdx$

Ans: $\int \cos(sinx)\ cosxdx$

$let\ sinx = t \Rightarrow \frac{d}{dx}(sinx) = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$

$= \int cost\ dt = sint + c = \sin(sinx) + c$

Ex: $\int e^{tanx} sec^2x\ dx$

Ans: $\int e^{tanx} sec^2x\ dx$

Let $tanx = t \Rightarrow sec^2xdx = dt$

so $\int e^{tanx} sec^2x\ dx = \int e^t dt = e^t + c = e^{tanx} + c$

Ex: $\int x^2 e^{x^3}\, dx$

Ans: $\int x^2 e^{x^3}\, dx$

Let $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = dt/3$

so$\int x^2 e^{x^3}\, dx = \int \frac{e^t dt}{3} = \frac{1}{3}e^t + c = \frac{1}{3}e^{x^3} + c$

Ex:$\int a^{sinx} cosx\, dx$

Ans: $\int a^{sinx} cosx\, dx$

Let $sinx = t \Rightarrow cosxdx = dt$

$=\int a^t dt = \frac{a^t}{lna} + c$

Ex:$\int \frac{b^{lnx}}{x} dx$

Let $lnx = t \Rightarrow \frac{1}{x}dx = dt$

$=\int b^t dt = \frac{b^t}{lnb} + c$

2. $\int \frac{f'(x)}{\left(f(x)\right)^n} dx$

How to solve:

Let f(x) = t, take derivative in both sides

$\Rightarrow \frac{d}{dx}f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$

So $\int \frac{f'(x)}{\left(f(x)\right)^n} dx\ = \int \frac{dt}{t^n} = \int t^{-n} dt = \frac{t^{-n+1}}{-n+1} + c = \frac{(f(x))^{1-n}}{1-n} + c$

Ex: Evaluate $\int \frac{sec^2 x}{tanx} dx$

Ans: $\int \frac{sec^2 x}{tanx} dx$

Let $tanx = t \Rightarrow \frac{d}{dx}tanx = \frac{dt}{dx} \Rightarrow sec^2 xdx = dt$

$= \int \frac{dt}{t} = ln|t| + c = ln|tanx| + c$

Ex: Evaluate $\int \frac{sec^2x}{tan^3x}dx$

Ans: $\int \frac{sec^2x}{tan^3x}dx$

Let $tanx = t \Rightarrow \frac{d}{dx}tanx = \frac{dt}{dx} \Rightarrow sec^2xdx = dt$

$= \int \frac{dt}{t^3} = \int t^{-3}dt = \frac{t^{-2}}{-2} + c = -\frac{1}{2}(tanx)^{-2} + c$

Ex:$\int \frac{cosx}{3+4sinx}dx$

Let $3 + 4sinx = t \Rightarrow 4cosxdx = dt \Rightarrow cosxdx = \frac{dt}{4}$

=$\int \frac{dt/4}{t} = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}ln|t| + c = \frac{1}{4}ln|3 + 4sinx| + c$

Ex:$\int \frac{x}{a^2+x^2}dx$

Let $a^2 + x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = dt/2$

=$\int \frac{dt/2}{t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}ln|t| + c$

Ex:$\int \frac{x}{\sqrt{a^2+x^2}}dx$

Let $a^2 + x^2 = t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

=$\int \frac{tdt}{t} = \int dt = t + c = \sqrt{a^2 + x^2} + c$

Ex:$\int \frac{sec^2\sqrt{x}}{\sqrt{x}}dx$

Let $\sqrt{x}$=t $\Rightarrow \frac{1}{2\sqrt{x}}$dx=dt $\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

=$\int sec^2t\ 2dt = 2\int sec^2t\,dt = 2tant + c = 2tan\sqrt{x} + c$

Ex:$\int \frac{\left(sin^{-1}x\right)^2}{\sqrt{1-x^2}}dx$

let $sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

=$\int \frac{\left(sin^{-1}x\right)^2}{\sqrt{1-x^2}}dx = \int t^2dt = \frac{t^3}{3} + c$

Some formulas related to substitutions

F-1. $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+c$

Proof: Let $ax+b=t$

$\Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a}$

$so \int f(ax+b)dx = \int f(t)\frac{dt}{a} = \frac{1}{a}F(t)+c = \frac{1}{a}F(ax+b)+c$

F-2. $\int sin(ax+b)dx = -\frac{1}{a}cos(ax+b)+c$

Ex: $\int sin(2x+4)dx = -\frac{1}{2}cos(2x+4)+c$ $\{or\ we\ can\ substitute\ 2x+4=t$

Ex: $\int sin(2x)dx = -\frac{1}{2}cos(2x)+c$

Ex: $\int \sin(2-7x)\,dx = \frac{-cos(2-7x)}{-7}+c$

Ex: $\int sin\left(\frac{x}{3}\right)dx = -\frac{cos\frac{x}{3}}{\frac{1}{3}}+c = -3cos\left(\frac{x}{3}\right)+c$

F-3. $\int cos(ax+b)dx = \frac{1}{a}sin(ax+b)+c$

Ex: $\int cos4x\,dx = \frac{sin4x}{4}+c$

Ex: $\int \cos(2-x)\,dx = \frac{\sin(2-x)}{-1}+c$

F-4. $\int sec^2(ax+b)dx = \frac{1}{a}tan(ax+b)+c$

Ex: $\int sec^2 4x\,dx = \frac{tan4x}{4}+c$

Ex: $\int sec^2(2+x)dx = \tan(2+x)+c$

F-5. $\int cosec^2(ax+b)dx = -\frac{1}{a}cot(ax+b)+c$

Ex: $\int cosec^2 3x\,dx = -\frac{cot3x}{3}+c$

Ex: $\int cosec^2 7x\,dx = -$ cot7x/7+c

F-6. $\int sec(ax+b)tan(ax+b)dx = \frac{1}{a}sec(ax+b)+c$

Ex: $\int \sec(2x+1)\tan(2x+1)\,dx = \frac{\sec(2x+1)}{2} + c$

Ex: $\int \sec(x+1)\tan(x+1)\,dx = \sec(x+1) + c$

F-7. $\int cosec(ax+b)cot(ax+b)dx = -\frac{1}{a}cosec(ax+b) + c$

Ex: $\int \operatorname{cosec}(2x+1)\cot(2x+1)\,dx = -\frac{cosec(2x+1)}{2} + c$

Ex: $\int \operatorname{cosec}(x+1)\cot(x+1)\,dx = -cosec(x+1) + c$

F-8. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

Ex: $\int e^{2x+3}dx = \frac{e^{2x+3}}{2} + c$

Ex: $\int e^{-x}dx = \frac{e^{-x}}{-1} + c = -e^{-x} + c$

Ex: $\int e^{-2x}dx = \frac{e^{-2x}}{-2} + c$

F-9. $\int a^{bx+d}dx = \frac{1}{b}\frac{a^{bx+d}}{lna} + c$

F-10. $\int \frac{1}{ax+b}dx = \frac{1}{a}ln|ax+b| + c$

Ex: $\int \frac{1}{3x-2}dx = \frac{ln|3x-2|}{3} + c$

Ex: $\int \frac{1}{2-7x}dx = \frac{ln|2-7x|}{-7} + c$ (IMP)

Ex: $\int \frac{1}{2-x}dx = \frac{ln|2-x|}{-1} + c$

F-11. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c, n \neq -1$

Ex: $\int (2x+1)^{11}dx = \frac{1}{2}\frac{(2x+1)^{12}}{12} + c$

F-12. $\int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}tan^{-1}(ax+b) + c$

Ex: $\int \frac{1}{1+(2x+1)^2}dx = \frac{1}{2}tan^{-1}(2x+1) + c$

F-13. $\int \frac{1}{\sqrt{1-(ax+b)^2}}dx = \frac{1}{a}sin^{-1}(ax+b) + c$

Ex: $\int \frac{1}{\sqrt{1-(2x+1)^2}} = \frac{1}{2} sin^{-1}(2x+1) + c$

F-14. $\int \frac{1}{(ax+b)\sqrt{(ax+b)^2-1}} dx = \frac{1}{a} sec^{-1}(ax+b) + c$

F-15. $\int tanx\, dx = ln|secx| + c$

F-16. $\int cotx\, dx = ln|sinx| + c$

F-17. $\int secx\, dx = ln|secx + tanx| + c$

F-18. $\int cosecx\, dx = ln|cosecx - cotx| + c$

Ex: Evaluate $\int tan7x\, dx$

Ans: $\int tan7x\, dx$

$let\ 7x = t \Rightarrow 7\, dx = dt \Rightarrow dx = \frac{dt}{7}$

$= \int tant\, \frac{dt}{7} = \frac{1}{7} \int tant\, dt = \frac{1}{7} ln|sect| + c = \frac{1}{7} ln|sec7x| + c$

Ex: Evaluate $\int cot7x\, dx$

Ans: $\int cot7x\, dx$

$let\ 7x = t \Rightarrow 7\, dx = dt \Rightarrow dx = \frac{dt}{7}$

$= \int cott\, \frac{dt}{7} = \frac{1}{7} \int cott\, dt = \frac{1}{7} ln|sint| + c = \frac{1}{7} ln|sin7x| + c$

Ex: Evaluate $\int \sec(2x+1)\, dx$

Ans: $\int \sec(2x+1)\, dx$

$let\ 2x + 1 = t \Rightarrow 2\, dx = dt \Rightarrow dx = \frac{dt}{2}$

$= \int sect\, \frac{dt}{2} = \frac{1}{2} \int sect\, dt = \frac{1}{2} ln|sect + tant| + c$

$= \frac{1}{2} ln|\sec(2x+1) + \tan(2x+1)| + c$

Ex: Evaluate $\int \text{cosec}(2x-3)\, dx$

Ans: $\int \text{cosec}(2x-3)\, dx$

$let\ 2x - 3 = t \Rightarrow 2\, dx = dt \Rightarrow dx = \frac{dt}{2}$

$=\int cosect\ \frac{dt}{2} = \frac{1}{2}\int cosect\ dt = \frac{1}{2} ln|cosect - cott| + c$

$$= \frac{1}{2} ln|\mathrm{cosec}(2x-3) - \cot(2x-3)| + c$$

IMP:

Ex: $\int \frac{sinx}{\sin(x-\alpha)} dx$

Ans: $\int \frac{sinx}{\sin(x-\alpha)} dx$

$let\ \ x - \alpha = t \Rightarrow dx = dt \qquad (x = t + \alpha)$

$=\int \frac{\sin(t+\alpha)}{\sin t} dt = \int \frac{\mathrm{sint\ cos\alpha + cost\ sin\alpha}}{\mathrm{sin}t} dt = \int \left(\frac{\mathrm{sint\ cos\alpha}}{\mathrm{sin}t} + \frac{\mathrm{cost\ sin\alpha}}{sint}\right) dt$

$=\int cos\alpha\ dt + \int sin\alpha\ cott\ dt = cos\alpha\ \ t + sin\alpha\ \ ln|sint| + c$

$=cos\alpha\ (x-\alpha) + sin\alpha\ \ ln|\sin(x-\alpha)| + c$

Assignment:

Ex: $\int \frac{sinx}{\sin(x+\alpha)} dx$

Ex: $\int \frac{cosx}{\sin(x+\alpha)} dx$

Ex: $\int \frac{cosx}{\cos(x+\alpha)} dx$

Ex: $\int \frac{sinx}{\cos(x-\alpha)} dx$

Ex: $\int \tan(x+1)\, dx$

Ex: $\int \cot(2x-11)\, dx$

Ex: $\int \sec(3x)\, dx$

Ex: $\int \mathrm{cosec}\, 7\mathrm{x}\, dx$

Ex: $\int \sec^2(2x+1)\, dx$

Ex: $\int \sin(2x-11)\, dx$

Ex: $\int \sec(3x) tan3x\, dx$

Ex: $\int e^{2x-7} dx$

Ex: $\int \frac{1}{2-3x} dx$

IMP:

$EX: \int \frac{e^x+e^{-x}}{e^x-e^{-x}} dx$

Ex: $\int \frac{tanx+tan\alpha}{tanx-tan\alpha} dx$

Ex: $\int cotx\sqrt{\ln sinx}\, dx$

Ex: $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$

Ex: $\int \frac{e^{tanx}}{cos^2x} dx$

Ex: $\int \frac{1}{sinx\ cosx} dx$

Ex: $\int \frac{1}{1-e^{-x}} dx$, $\int \frac{1}{1+e^{-x}} dx$, $\int \frac{1}{1+e^x} dx$

Ex: $\int \frac{sin2x}{acos^2x+bsin^2x} dx$

Ex: $\int \frac{cosec^2x}{3-cotx} dx$

Ex: $\int \frac{sinx}{cos^9x} dx$

Ex: $\int \frac{cosx}{\sqrt{1-sinx}} dx$

Ex: $\int \frac{x}{\sqrt{x^2-a^2}} dx$

Ex: $\int sinxe^{cosx} dx$

Ex: $\int x\sqrt{a^2+x^2} dx$

Ex: $\int cosx\ cos(sinx) dx \quad let\ sinx = t$

INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS

$$sin^2x = 1 - cos^2x \text{ , } cos^2x = 1 - sin^2x \text{ , } sec^2x = 1 + tan^2x \text{ ,}$$

$$cosec^2x = 1 + cot^2x \text{ , } cos^2x = \frac{1+cos2x}{2}, sin^2x = \frac{1-cos2x}{2}$$

$$Sin(A+B) + sin(A-B) = 2sinAcosB$$

$$Sin(A+B) - Sin(A-B) = 2cosAsinB$$

$$cos(A+B) + cos(A-B) = 2cosAcosB$$

$$\begin{cases} cos(A+B) - cos(A-B) = -2sinAsinB \\ or\ cos(A-B) - cos(A+B) = 2sinAsinB \end{cases}$$

Ex: $\int sin\ 3x\ cos2xdx$

Ans: $\int sin\ 3x\ cos2xdx = \frac{1}{2}\int 2\ sin\ 3x\ cos2xdx$

$= \frac{1}{2}\int(\sin(3x+2x) + \sin(3x-2x))dx$

$= \frac{1}{2}\int sin5xdx + \frac{1}{2}\int sinx\ dx$

$= \frac{1}{2}\left(\frac{-cos5x}{5}\right) + \frac{1}{2}(-cosx) + c$

$= -\frac{cos5x}{10} - \frac{cosx}{2} + c$

Ex: $\int cos5x\ cos2xdx$

Ans: $\int cos\ 5x\ cos2xdx = \frac{1}{2}\int 2\ cos5x\ cos2xdx$

$= \frac{1}{2}\int(\cos(5x+2x) + \cos(5x-2x))dx$

$= \frac{1}{2}\int cos7xdx + \frac{1}{2}\int cos3x\ dx$

$= \frac{1}{2}\left(\frac{sin7x}{7}\right) + \frac{1}{2}\left(\frac{sin3x}{3}\right) + c$

$= \frac{sin7x}{14} + \frac{sin3x}{6} + c$

Ex: $\int \boldsymbol{cos3x\ sinx\ sin5x\ dx}$

Ans: $\int cos3x\ sinx\ sin5xdx$

$= \frac{1}{2}\int (2sin5xcos3x)sinx\ dx$

$= \frac{1}{2}\int (sin8x + sin2x)sinxdx$

$= \frac{1}{2}\int sin8xsinx\ dx + \frac{1}{2}\int sin2xsinx\ dx$

$= \frac{1}{4}\int 2sin8x\ sinx\ dx + \frac{1}{4}\int 2sin2x\ sinx\ dx$

$$= \frac{1}{4}\int (\cos(8x - x) - \cos(8x + x))\, dx + \frac{1}{4}\int (\cos(2x - x) - \cos(2x + x))\, dx$$

$= \frac{1}{4}\int \cos 7x\, dx - \frac{1}{4}\int \cos 9x\, dx + \frac{1}{4}\int \cos x\, dx - \frac{1}{4}\int cos3xdx$

$= \frac{1}{4}\frac{sin7x}{7} - \frac{1}{4}\frac{sin9x}{9} + \frac{1}{4}sinx - \frac{1}{4}\frac{sin3x}{3} + c$

<u>Here the power of sinx and cosx are odd</u>

Ex: $\int \boldsymbol{sin^3x\ dx}$

Ans: $\int sin^3x\ dx = \int sin^2x\ sinxdx = \int (1 - cos^2x)sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$= \int (1 - t^2)(-dt) = \int (t^2 - 1)dt = \int t^2dt - \int dt = \frac{t^3}{3} - t + c$

$= \frac{cos^3x}{3} - cosx + c$

Ex: $\int \boldsymbol{sin^5x\ dx}$

Ans: $\int sin^5x\ dx = \int sin^4x\ sinxdx = \int (sin^2x)^2 sinxdx = \int (1 - cos^2x)^2 sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$= \int (1 - t^2)^2(-dt) = \int (1 + t^4 - 2t^2)(-dt) = \int (2t^2 - t^4 - 1)dt$

$= \int 2t^2dt - \int t^4dt - \int dt = \frac{2t^3}{3} - \frac{t^5}{5} - t + c$

$=\frac{2cos^3x}{3}-\frac{cos^5x}{5}-cosx+c$

Ex: $\int \boldsymbol{sin^7x\,dx}$

Ans: $\int sin^7x\,dx=\int sin^6x\,sinxdx=\int (sin^2x)^3 sinxdx=\int (1-cos^2x)^3 sinxdx$

Let $cosx=t \Rightarrow -sinxdx=dt$

$=\int (1-t^2)^3(-dt)=\int (1-t^6-3t^2+3t^4)(-dt)=\int (t^6-3t^4+3t^2-1)dt$

$=\int t^6dt-\int 3t^4dt+\int 3t^2dt-\int dt=\frac{t^6}{6}-3\frac{t^5}{5}+3\frac{t^3}{3}-t+c$

$=\frac{cos^6x}{6}-3\frac{cos^5x}{5}+3\frac{cos^3x}{3}-cosx+c$ USE: $(a-b)^3=a^3-b^3-3a^2b+3ab^2$

Ex: $\int sin^5x\,cos^3x$dx

Ans: $\int sin^5x\,cos^3xdx=\int sin^5x\,cos^2x\,cosxdx$

$=\int sin^5x\,(1-sin^2x)\,cosxdx$

Let $sinx=t \Rightarrow cosxdx=dt$

$=\int t^5\,(1-t^2)dt=\int t^5dt-\int t^7dt=\frac{t^6}{6}-\frac{t^8}{8}+c=\frac{sin^6x}{6}-\frac{sin^8x}{8}+c$

Ex: $\int sin^3x\,cos^8x$dx

Ans: $\int sin^3x\,cos^8xdx=\int cos^8x\,sin^3xdx=\int cos^8x\,sin^2x\,sinxdx$

$=\int cos^8x\,(1-cos^2x)\,sinxdx$

Let $cosx=t \Rightarrow -sinxdx=dt$

$=\int t^8\,(1-t^2)(-dt)=\int (t^8-t^{10})(-dt)$

$=\int -t^8dt+\int t^{10}dt=-\frac{t^9}{9}+\frac{t^{11}}{11}+c=-\frac{cos^9x}{9}+\frac{cos^{11}x}{11}+c$

Ex: $\int \frac{sin^3x}{cos^9x}dx$

Ans: $\int \frac{sin^3x}{cos^9x}dx=\int \frac{sin^2xsinx}{cos^9x}dx=\int \frac{(1-cos^2x)sinx}{cos^9x}dx$

Let $cosx=t \Rightarrow -sinxdx=dt$

$=\int \frac{(1-t^2)}{t^9}(-dt)=\int \frac{(t^2-1)}{t^9}dt=\int \frac{t^2}{t^9}dt-\int \frac{1}{t^9}dt=\int t^{-7}dt-\int t^{-9}dt$

$= \frac{t^{-6}}{-6} - \frac{t^{-8}}{-8} + c = -\frac{(cosx)^{-6}}{6} + \frac{(cosx)^{-9}}{9} + c$

Ex: $\int tan^2 xdx$

Ans: $\int tan^2 xdx = \int (sec^2 x - 1)dx = \int sec^2 xdx - \int 1\, dx = tanx - x + c$

Ex: $\int tan^4 xdx$

Ans: $\int tan^4 xdx = \int tan^2 x\, tan^2 xdx = \int tan^2 x(sec^2 x - 1)dx$

$= \int tan^2 x\, sec^2 xdx - \int tan^2 xdx$

Let $tanx = t \Rightarrow sec^2 xdx = dt$

$= \int t^2 dt - \int (sec^2 x - 1)dx = \int t^2 dt - \int sec^2 xdx + \int 1\, dx = \frac{t^3}{3} - tanx + x + c$

$= \frac{tan^3 x}{3} - tanx + x + c$

Ex: $\int tan^6 xdx$

Ans: $\int tan^6 xdx = \int tan^4 x tan^2 xdx = \int tan^4 x(sec^2 x - 1)dx$

$= \int tan^4 x sec^2 xdx - \int tan^4 xdx$

Let $tanx = t \Rightarrow sec^2 xdx = dt$

$= \int t^4 dt - \int tan^2 x(sec^2 x - 1)dx$

$= \int t^4 dt - \int tan^2 x sec^2 xdx + \int tan^2 x\, dx$

$= \int t^4 dt - \int t^2 dt + \int (sec^2 x - 1)dx$

$= \int t^4 dt - \int t^2 dt + \int sec^2 xdx - \int 1\, dx$

$= \frac{t^5}{5} - \frac{t^3}{3} + tanx - x + c = \frac{tan^5 x}{5} - \frac{tan^3 x}{3} + tanx - x + c$

Ex: $\int cot^3 xdx$

Ans: $\int cot^3 xdx = \int cotx\, cot^2 xdx = \int cotx(cosec^2 x - 1)dx$

$= \int cotx\, cosec^2 xdx - \int cotxdx$

Let $cotx = t \Rightarrow -cosec^2 xdx = dt$

$= \int t(-dt) - \int cotx\, dx = -\frac{t^2}{2} - ln|sinx| + c$

$=-\frac{cot^2x}{2}-ln|sinx|+c$

Note: The power of tan is either even or odd but the power of sec is even

Ex: $\int tan^5x\, sec^2x\, dx$

Ans: $\int tan^5x\, sec^2x\, dx$

Let $tanx = t \Rightarrow\ sec^2xdx = dt$

$=\int t^5dt = \frac{t^6}{6}+c = \frac{tan^6x}{6}+c$

Ex: $\int tan^{10}x\, sec^4x\, dx$

Ans: $\int tan^{10}x\, sec^4x\, dx = \int tan^{10}x\, sec^2x\ \ sec^2xdx$

$=\int tan^{10}x\ \ (1+tan^2x)sec^2xdx$

Let $tanx = t \Rightarrow\ sec^2xdx = dt$

$=\int t^{10}(1+t^2)dt = \int t^{10}dt + \int t^{12}dt = \frac{t^{11}}{11}+\frac{t^{13}}{13}+c$

$=\frac{tan^{11}x}{11}+\frac{tan^{13}x}{13}+c$

Note: The power of sec is either odd or even but the power of tan is odd

Ex: $\int sec^{11}x\, tanx\, dx$

Ans: $\int sec^{11}x\, tanx\, dx = \int sec^{10}x\, secx\, tanx\, dx$

Let $secx = t \Rightarrow secx\, tanx\, dx = dt$

$=\int t^{10}dt = \frac{t^{11}}{11}+c = \frac{sec^{11}x}{11}+c$

Ex: $\int sec^{12}x\, tan^3x\, dx$

Ans: $\int sec^{12}x\, tan^3x\, dx = \int sec^{11}x\, tan^2x\, secx\, tanx\, dx$

$=\int sec^{11}x\, (sec^2x-1)\, secx\, tanx\, dx$

Let $secx = t \Rightarrow secx\, tanx\, dx = dt$

$= \int t^{11}(t^2 - 1)dt = \int t^{13}dt - \int t^{11}dt = \frac{t^{14}}{14} - \frac{t^{12}}{12} + c$

$= \frac{sec^{14}x}{14} - \frac{sec^{12}x}{12} + c$

Ex: $\int sin^2x\, dx$

Ans: $\int sin^2x\, dx = \int \frac{1-cos2x}{2} dx = \int \frac{1}{2} dx - \int \frac{cos2x}{2} dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int cos2x\, dx = \frac{1}{2}x - \frac{1}{2}\frac{sin2x}{2} + c$

Ex: $\int cos^2x\, dx$

Ans: $\int cos^2x\, dx = \int \frac{1+ cos2x}{2} dx = \int \frac{1}{2} dx + \int \frac{cos2x}{2} dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int cos2x\, dx = \frac{1}{2}x + \frac{1}{2}\frac{sin2x}{2} + c$

Ex: $\int sin^2 2x\, dx = \int \frac{1-cos4x}{2} dx = \int \frac{1}{2} dx - \int \frac{cos4x}{2} dx$ (Hints)

Ex: $\int cos^2 2x\, dx = \int \frac{1+cos4x}{2} dx = \int \frac{1}{2} dx + \int \frac{cos4x}{2} dx$ (Hints)

Ex: $\int sin^4x\, dx$

Ans: $\int sin^4x\, dx = \int (sin^2x)^2\, dx = \int \left(\frac{1-cos2x}{2}\right)^2 dx$

$= \frac{1}{4}\int (1 + cos^2 2x - 2cos2x)dx = \frac{1}{4}\int \left(1 + \frac{1+cos4x}{2} - 2cos2x\right) dx$

$= \frac{1}{4}\int \left(\frac{2+1+cos4x-4cos2x}{2}\right) dx = \frac{1}{8}\int (3 + cos4x - 4cos2x)dx$

$= \frac{1}{8}\int 3dx + \frac{1}{8}\int cos4xdx - \frac{1}{8}\int 4cos2x\, dx = \frac{3}{8}x + \frac{1}{8}\frac{sin4x}{4} - \frac{1}{2}\frac{sin2x}{2} + c$

Assignment:

Ex.$\int sin3x\ cos2x\ dx$

Ex.$\int cos5x\ sin3x\ dx$

Ex.$\int cos4x\ cos2x\ dx$

Ex.$\int sin3x\ sin4x\ dx$

Ex.$\int cos2x\ cos4x\ dx$

Ex.$\int sinx\ sin2x\ sin3x\ dx$

Ex.$\int cos^3 x\ dx$

Ex.$\int cos^5 x\ dx$

Ex.$\int cos^7 x\ dx$

Ex.$\int sin^5 xcos^2 x\ dx$

Ex.$\int sin^4 xcosx\ dx$

Ex.$\int sin^3 xcos^2 x\ dx$

Ex.$\int \frac{sin^3 x}{cosx} dx$

Ex.$\int cosmx\ cosnx\ dx$

Ex.$\int \frac{cos4x - cos2x}{sin4x - sin2x} dx$

Ex.$\int sec^n x\ tanx\ dx$

Ex:$\int sin^5 xcos^2 x\ dx$

Ex:$\int sin^4 xcosx\ dx$

Ex:$\int sin^3 xcos^2 x\ dx$

Ex:$\int sin^3 xcos^3 x\ dx$

Ex:$\int cot^2 x\ dx$

Ex:$\int cot^4 x\ dx$

Ex: $\int cot^7 x\ dx$

Ex: $\int tan^3 x\ dx$

Ex: $\int tan^5 x\ dx$

Ex:$\int cot^{11} x\ cosec^2 x\ dx$

Ex:$\int cot^{14} x\ cosec^4 x\ dx$

Ex: $\int sec^{12} x\ tan^3 x\ dx$

Ex: $\int cosec^{14} x\ cot^3 x\ dx$

Ex: $\int cosec^{12} x\ cot^3 x\ dx$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

F-1:If the integral in the form of $\int \frac{dx}{\sqrt{a^2-x^2}}$ then substitute $x = a\sin\theta$, $\sin\theta = x/a$

Proof: $\int \frac{dx}{\sqrt{a^2-x^2}}$ put $x = a\sin\theta \Longrightarrow dx = a\cos\theta\, d\theta$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\frac{x}{a} + c$$

Formula: $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$

Ex: $\int \frac{dx}{\sqrt{9-x^2}}$

Ans: $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = sin^{-1}\frac{x}{3} + c$

Ex: $\int \frac{dx}{\sqrt{7-9x^2}}$

Ans: $\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}}$

Let $3x = t \Longrightarrow 3dx = dt \Longrightarrow dx = dt/3$

$$= \int \frac{dt/3}{\sqrt{(\sqrt{7})^2-t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{(\sqrt{7})^2-t^2}} = \frac{1}{3}sin^{-1}\frac{t}{\sqrt{7}} + c = \frac{1}{3}sin^{-1}\frac{3x}{\sqrt{7}} + c$$

Ex: $\int \frac{xdx}{\sqrt{9-x^4}}$

Ans: $\int \frac{xdx}{\sqrt{9-x^4}} = \int \frac{xdx}{\sqrt{3^2-(x^2)^2}}$

Let $x^2 = t \Longrightarrow 2xdx = dt \Longrightarrow xdx = dt/2$

$$=\int \frac{dt/2}{\sqrt{3^2-t^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{3^2-t^2}} = \frac{1}{2}sin^{-1}\frac{t}{3} + c = \frac{1}{2}sin^{-1}\frac{x^2}{3} + c$$

Ex: $\int \frac{e^x dx}{\sqrt{9-e^{2x}}}$

Ans: $\int \frac{e^x dx}{\sqrt{9-e^{2x}}} = \int \frac{e^x dx}{\sqrt{3^2-(e^x)^2}}$

Let $e^x = t \Rightarrow e^x dx = dt$

$=\int \frac{dt}{\sqrt{3^2-t^2}} = sin^{-1}\frac{t}{3} + c = sin^{-1}\frac{e^x}{3} + c$

Ex: $\int \frac{cosx\, dx}{\sqrt{16-sin^2x}}$

Ans: $\int \frac{cosx\, dx}{\sqrt{16-sin^2x}} = \int \frac{cosx\, dx}{\sqrt{4^2-sin^2x}}$

Let $sinx = t \Rightarrow cosx\, dx = dt$

$=\int \frac{dt}{\sqrt{4^2-t^2}} = sin^{-1}\frac{t}{4} + c = sin^{-1}\frac{sinx}{4} + c$

Ex: $\int \frac{dx}{x\sqrt{25-(lnx)^2}}$

Ans: $\int \frac{dx}{x\sqrt{25-(lnx)^2}}$

$= \int \frac{dx}{x\sqrt{5^2-(lnx)^2}}$

Let $lnx = t \Rightarrow \frac{1}{x} dx = dt$

$=\int \frac{dt}{\sqrt{5^2-t^2}} = sin^{-1}\frac{t}{5} + c = sin^{-1}\frac{lnx}{5} + c$

F-2: If the integral in the form of $\int \frac{dx}{a^2+x^2}$ then substitute $x = a\tan\theta, \tan\theta = x/a$

Proof: $\int \frac{dx}{a^2+x^2}$

Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Formula: $\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1} \frac{x}{a} + c$

Ex: $\int \frac{dx}{9+x^2}$

Ans: $\int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2} = \frac{1}{3} tan^{-1} \frac{x}{3} + c$

Ex: $\int \frac{dx}{11+16x^2}$

Ans: $\int \frac{dx}{11+16x^2} = \int \frac{dx}{(\sqrt{11})^2+(4x)^2}$

Let $4x = t \Rightarrow 4\,dx = dt \Rightarrow dx = dt/4$

$= \int \frac{dt/4}{(\sqrt{11})^2+(t)^2} = \frac{1}{4}\int \frac{dt}{(\sqrt{11})^2+(t)^2} = \frac{1}{4}\frac{1}{\sqrt{11}} tan^{-1} \frac{t}{\sqrt{11}} + c = \frac{1}{4\sqrt{11}} tan^{-1} \frac{4x}{\sqrt{11}} + c$

Ex: $\int \frac{x^2}{16+x^6} dx$

Ans: $\int \frac{x^2}{16+x^6} dx = \int \frac{x^2}{4^2+(x^3)^2} dx$

Let $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = dt/3$

$=\int \frac{dt/3}{4^2+(t)^2} = \frac{1}{3}\int \frac{dt}{4^2+(t)^2} = \frac{1}{3}\frac{1}{4} tan^{-1} \frac{t}{4} + c = \frac{1}{12} tan^{-1} \frac{x^3}{4} + c$

Ex: $\int \frac{e^{2x}dx}{9+e^{4x}}$

Ans: $\int \frac{e^{2x}dx}{9+e^{4x}} = \int \frac{e^{2x}dx}{3^2+(e^{2x})^2}$

Let $e^{2x} = t \Rightarrow e^{2x}\, 2dx = dt \Rightarrow e^{2x}\, dx = dt/2$

$=\int \frac{dt/2}{3^2+t^2} = \frac{1}{2}\int \frac{dt}{3^2+t^2} = \frac{1}{2}\ \frac{1}{3} tan^{-1} \frac{t}{3} + c = \frac{1}{6} tan^{-1} \frac{e^{2x}}{3} + c$

Ex: $\int \frac{sec^2x\, dx}{16+tan^2x}$

Ans: $\int \frac{sec^2xdx}{16+tan^2x} = \int \frac{sec^2xdx}{4^2+tan^2x}$

Let $tanx = t \Rightarrow \sec^2 x\, dx = dt$

$= \int \frac{dt}{4^2+t^2} = \frac{1}{4} tan^{-1} \frac{tanx}{4} + c$

Ex: $\int \frac{dx}{x(25+(lnx)^2)}$

Ans: $\int \frac{dx}{x(25+(lnx)^2)} = \int \frac{dx}{x(5^2+(lnx)^2)}$

Let $lnx = t \Rightarrow \frac{1}{x} dx = dt$

$= \int \frac{dt}{5^2+t^2} = \frac{1}{5} tan^{-1} \frac{t}{5} + c = \frac{1}{5} tan^{-1} \frac{lnx}{5} + c$

F-3: If the integral in the form of $\int \frac{dx}{\sqrt{a^2+x^2}}$ then substitute $x = a\tan\theta$

Proof: $\int \frac{dx}{\sqrt{x^2+a^2}}$

Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$

$$= \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \frac{a\sec^2\theta d\theta}{a\sqrt{\tan^2\theta + 1}} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\frac{x}{a} + \sqrt{\tan^2\theta + 1}\right| + c_1 = \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right| + c_1 = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 + a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c_1 = \ln\left|x + \sqrt{x^2 + a^2}\right| - \ln a + c_1 = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{9+x^2}}$

Ans: $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{dx}{\sqrt{3^2+x^2}} = ln\left|x + \sqrt{3^2 + x^2}\right| + c = ln\left|x + \sqrt{9 + x^2}\right| + c$

Ex: $\int \frac{dx}{\sqrt{7+9x^2}}$

Ans: $\int \frac{dx}{\sqrt{7+9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2+(3x)^2}}$

Let $3x = t \Rightarrow 3dx = dt \Rightarrow dx = dt/3$

$= \int \frac{dt/3}{\sqrt{(\sqrt{7})^2+t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{(\sqrt{7})^2+t^2}} = \frac{1}{3} ln\left|t + \sqrt{(\sqrt{7})^2 + t^2}\right| + c$

$= \frac{1}{3} ln\left|3x + \sqrt{7 + 9x^2}\right| + c$

Ex: $\int \frac{x^4dx}{\sqrt{9+x^{10}}}$

Ans: $\int \frac{x^4dx}{\sqrt{9+x^{10}}} = \int \frac{x^4dx}{\sqrt{3^2+(x^5)^2}}$

Let $x^5 = t \Rightarrow 5x^4dx = dt \Rightarrow x^4dx = dt/5$

$= \int \frac{dt/5}{\sqrt{3^2+t^2}} = \frac{1}{5}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{5} ln\left|t + \sqrt{(3)^2 + t^2}\right| + c = \frac{1}{5} ln\left|x^5 + \sqrt{9 + x^{10}}\right| + c$

Ex: $\int \frac{a^xdx}{\sqrt{9+a^{2x}}}$

Ans: $\int \frac{a^xdx}{\sqrt{9+a^{2x}}} = \int \frac{a^xdx}{\sqrt{3^2+(a^x)^2}}$

Let $a^x = t \Rightarrow a^x \ln a\, dx = dt \Rightarrow a^x dx = dt/\ln a$

$= \int \frac{\frac{dt}{lna}}{\sqrt{3^2+t^2}} = \frac{1}{lna}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{lna} ln\left|t + \sqrt{(3)^2 + t^2}\right| + c$

$= \frac{1}{lna} ln\left|a^x + \sqrt{9 + a^{2x}}\right| + c$

Ex: $\int \frac{cosec^2x\, dx}{\sqrt{16+cot^2x}}$

Ans: $\int \frac{cosec^2x\, dx}{\sqrt{16+cot^2x}} = \int \frac{cosec^2xdx}{\sqrt{4^2+cot^2x}}$

Let $cotx = t \Rightarrow -cosec^2xdx = dt \Rightarrow cosec^2xdx = -dt$

$= \int \frac{-dt}{\sqrt{4^2+t^2}} = -ln\left|t + \sqrt{(4)^2 + t^2}\right| + c = -ln\left|cotx + \sqrt{16 + cot^2x}\right| + c$

Ex: $\int \frac{dx}{x\sqrt{25+(lnx)^2}}$

Ans: $\int \frac{dx}{x\sqrt{25+(lnx)^2}}$

$= \int \frac{dx}{x\sqrt{5^2+(lnx)^2}}$

Let $lnx = t \Rightarrow \frac{1}{x}\,dx = dt$

$=\int \frac{dt}{\sqrt{5^2+t^2}} = ln\left|t + \sqrt{(5)^2 + t^2}\right| + c = ln\left|lnx + \sqrt{25 + (lnx)^2}\right| + c$

F-4: If an integral in the form of $\int \frac{dx}{\sqrt{x^2-a^2}}$ then substitute $x = a\sec\theta$

Proof: $\int \frac{dx}{\sqrt{x^2-a^2}}$

$$\text{Put } x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$$

$$= \int \frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sqrt{\sec^2\theta - 1}} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\frac{x}{a} + \sqrt{\sec^2\theta - 1}\right| + c_1 = \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right| + c_1 = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1 = \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + c_1 = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{x^2-9}}$

Ans: $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}} = ln\left|x + \sqrt{x^2 - 3^2}\right| + c = ln\left|x + \sqrt{x^2 - 9}\right| + c$

Ex: $\int \frac{dx}{\sqrt{9x^2-7}}$

Ans: $\int \frac{dx}{\sqrt{9x^2-7}} = \int \frac{dx}{\sqrt{(3x)^2-(\sqrt{7})^2}}$

Let $3x = t \Rightarrow 3dx = dt \Rightarrow dx = dt/3$

$= \int \frac{dt/3}{\sqrt{t^2-(\sqrt{7})^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{t^2-(\sqrt{7})^2}} = \frac{1}{3} ln\left|t + \sqrt{t^2 - (\sqrt{7})^2}\right| + c$

$= \frac{1}{3} ln\left|3x + \sqrt{9x^2 - 7}\right| + c$

Ex: $\int \frac{x^3 dx}{\sqrt{x^8-9}}$

Ans: $\int \frac{x^3 dx}{\sqrt{x^8-9}} = \int \frac{x^3 dx}{\sqrt{(x^4)^2-3^2}}$

Let $x^4 = t \Rightarrow 4x^3 \mathrm{dx} = \mathrm{dt} \Rightarrow x^3 \mathrm{dx} = \mathrm{dt}/4$

$= \int \frac{dt/4}{\sqrt{t^2-3^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{t^2-3^2}} = \frac{1}{4} ln\left|t + \sqrt{t^2 - 3^2}\right| + c = \frac{1}{4} ln\left|x^4 + \sqrt{x^8 - 9}\right| + c$

Ex: $\int \frac{e^x dx}{\sqrt{e^{2x}-11}}$

Ans: $\int \frac{e^x dx}{\sqrt{e^{2x}-11}} = \int \frac{e^x dx}{\sqrt{(e^x)^2-(\sqrt{11})^2}}$

Let $e^x = t \Rightarrow \mathrm{e^x\ dx = dt}$

$= \int \frac{dt}{\sqrt{t^2-(\sqrt{11})^2}} = ln\left|t + \sqrt{t^2 - (\sqrt{11})^2}\right| + c$

$= ln\left|e^x + \sqrt{e^{2x} - 11}\right| + c$

Ex: $\int \frac{dx}{sin^2x\sqrt{cot^2x-16}}$

Ans: $\int \frac{dx}{sin^2x\sqrt{cot^2x-16}} = \int \frac{cosec^2x dx}{\sqrt{cot^2x-4^2}}$

Let $cotx = t \Rightarrow -cosec^2x dx = \mathrm{dt} \Rightarrow cosec^2x dx = -\mathrm{dt}$

$= \int \frac{-dt}{\sqrt{t^2-4^2}} = -ln\left|t + \sqrt{t^2 - 4^2}\right| + c = -ln\left|cotx + \sqrt{cot^2x - 16}\right| + c$

Ex: $\int \frac{dx}{x\sqrt{(lnx)^2-25}}$

Ans: $\int \frac{dx}{x\sqrt{(lnx)^2-25}}$

$= \int \frac{dx}{x\sqrt{(lnx)^2-5^2}}$

Let $lnx = t \Rightarrow \frac{1}{x}\, dx = dt$

$=\int \frac{dt}{\sqrt{t^2-5^2}} = ln\left|t + \sqrt{t^2-5^2}\right| + c = ln\left|lnx + \sqrt{(lnx)^2 - 25}\right| + c$

IMPORTANT FORMS:

If the integral in the form $\int \frac{ax \pm b}{\sqrt{a^2 - x^2}} dx$ or $\int \frac{ax \pm b}{\sqrt{a^2 + x^2}} dx$ or $\int \frac{ax \pm b}{\sqrt{x^2 - a^2}} dx$ or $\int \frac{ax \pm b}{a^2 + x^2} dx$

then express $\int \frac{ax \pm b}{\sqrt{a^2 - x^2}} dx = \int \frac{ax}{\sqrt{a^2 - x^2}} dx \pm \int \frac{b}{\sqrt{a^2 - x^2}} dx$

Let $I_1 = \int \frac{ax}{\sqrt{a^2 - x^2}} dx$ and $I_2 = \int \frac{b}{\sqrt{a^2 - x^2}}$

Integrate I_1 by putting $a^2 - x^2 = t^2$ or $a^2 - x^2 = t$

Integrate I_2 by using the above formula .

Similarly we can integrate other forms.

Ex: $\int \frac{3x-2}{\sqrt{9-x^2}} dx$

Ans: $\int \frac{3x-2}{\sqrt{9-x^2}} dx = \int \frac{3x}{\sqrt{9-x^2}} dx - \int \frac{2}{\sqrt{9-x^2}} dx$

$= 3\int \frac{x}{\sqrt{9-x^2}} dx - 2\int \frac{dx}{\sqrt{3^2-x^2}}$

In the 1st term let $9 - x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$

$= 3\int \frac{-tdt}{t} - 2\int \frac{dx}{\sqrt{3^2-x^2}} = -3t - 2sin^{-1}\frac{x}{3} + c = -3\sqrt{9-x^2} - 2sin^{-1}\frac{x}{3} + c$

Ex: $\int \frac{x+2}{\sqrt{16+x^2}} dx$

Ans: $\int \frac{x+2}{\sqrt{16+x^2}} dx = \int \frac{x}{\sqrt{16+x^2}} dx + \int \frac{2}{\sqrt{16+x^2}} dx$

In the 1st term let $16 + x^2 = t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

$= \int \frac{tdt}{t} + 2\int \frac{dx}{\sqrt{4^2+x^2}} = t + 2\, ln\left|x + \sqrt{16+x^2}\right| + c$

$=\sqrt{16+x^2} + 2\, ln\left|x + \sqrt{16+x^2}\right| + c$

Ex: $\int \frac{2x+5}{7+x^2} dx$

Ans: $\int \frac{2x+5}{7+x^2} dx = \int \frac{2x}{7+x^2} dx + \int \frac{5}{7+x^2} dx$

Let $7 + x^2 = t \Rightarrow 2xdx = dt$

$= \int \frac{dt}{t} + 5\int \frac{dx}{(\sqrt{7})^2+x^2} = ln|t| + 5\frac{1}{\sqrt{7}} tan^{-1}\frac{x}{\sqrt{7}} + c$

$= ln|7 + x^2| + 5\frac{1}{\sqrt{7}} tan^{-1}\frac{x}{\sqrt{7}} + c$

Ex: $\int \frac{3x+1}{\sqrt{x^2-25}} dx$

Ans: $\int \frac{3x+1}{\sqrt{x^2-25}} dx = \int \frac{3x}{\sqrt{x^2-25}} dx + \int \frac{1}{\sqrt{x^2-25}} dx$

$= 3\int \frac{x}{\sqrt{x^2-25}} dx + \int \frac{1}{\sqrt{x^2-5^2}} dx$

In the 1st term let $x^2 - 25 = t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

$= 3\int \frac{tdt}{t} + \int \frac{1}{\sqrt{x^2-5^2}} dx = 3\,t + ln\left|x + \sqrt{x^2 - 5^2}\right| + c$

$= 3\sqrt{x^2 - 25} + ln\left|x + \sqrt{x^2 - 25}\right| + c$

IMP:

If the integral in the form $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

then express $ax^2 + bx + c = a\{(x-\alpha)^2 \pm \beta^2\}$ (In a perfect suare)

put $x - \alpha = t \Rightarrow dx = dt$

Ex: $\int \frac{dx}{x^2+4x+9}$

Ans: $\int \frac{dx}{x^2+4x+9}$

Make the denominator $x^2 + 4x + 9$ in a perfect square

Now $x^2 + 4x + 9 = x^2 + 2.x.2 + 2^2 - 2^2 + 9 = (x + 2)^2 + 5$

$:So \int \frac{dx}{x^2+4x+9} = \int \frac{dx}{(x+2)^2+5} = \int \frac{dx}{(x+2)^2+(\sqrt{5})^2}$

Let $x + 2 = t \Rightarrow dx = dt$

$$= \int \frac{dt}{t^2+\left(\sqrt{5}\right)^2} = \frac{1}{\sqrt{5}} tan^{-1} \frac{t}{\sqrt{5}} + c = \frac{1}{\sqrt{5}} tan^{-1} \frac{x+2}{\sqrt{5}} + c$$

Ex: $\int \frac{dx}{\sqrt{x^2-4x+13}}$

Ans: $\int \frac{dx}{\sqrt{x^2-4x+13}}$

Make the denominator $x^2 - 4x + 13$ in a perfect square

Now $x^2 - 4x + 13 = x^2 - 2.x.2 + 2^2 - 2^2 + 13 = (x-2)^2 + 9$

:$So \int \frac{dx}{\sqrt{x^2-4x+13}} = \int \frac{dx}{\sqrt{(x-2)^2+9}} = \int \frac{dx}{\sqrt{(x-2)^2+3^2}}$

Let $x - 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$$= \int \frac{dt}{\sqrt{t^2+3^2}} = ln\left|t + \sqrt{t^2 + 3^2}\right| + c = ln\left|(x-2) + \sqrt{(x-2)^2 + 3^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{x^2-4x-13}}$

Ans: $\int \frac{dx}{\sqrt{x^2-4x-13}}$

Now $x^2 - 4x - 13 = x^2 - 2.x.2 + 2^2 - 2^2 - 13 = (x-2)^2 - 17$

:$So \int \frac{dx}{\sqrt{x^2-4x-13}} = \int \frac{dx}{\sqrt{(x-2)^2-17}} = \int \frac{dx}{\sqrt{(x-2)^2-\left(\sqrt{17}\right)^2}}$

Let $x - 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$$= \int \frac{dt}{\sqrt{t^2-\left(\sqrt{17}\right)^2}} = ln\left|t + \sqrt{t^2 - \left(\sqrt{17}\right)^2}\right| + c$$

$$= ln\left|(x-2) + \sqrt{(x-2)^2 - \left(\sqrt{17}\right)^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{5-4x-x^2}}$

Ans: $\int \frac{dx}{\sqrt{5-4x-x^2}}$

Make the denominator $5 - 4x - x^2$ in a perfect square

Now $5-4x-x^2 = -(x^2+4x-5) = -(x^2+2.x.2+2^2-2^2-5)$

$$= -((x+2)^2-9) = 9-(x+2)^2$$

:So $\int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}} = \int \frac{dx}{\sqrt{3^2-(x+2)^2}}$

Let $x+2=t \Rightarrow \mathrm{dx}=\mathrm{dt}$

$=\int \frac{dt}{\sqrt{3^2-t^2}} = sin^{-1}\frac{t}{3}+c = sin^{-1}\frac{x+2}{3}+c$

If the integral in the form $\int \frac{(\mathrm{px}+\mathrm{q})\,\mathrm{dx}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}$ or $\int \frac{(\mathrm{px}+\mathrm{q})\mathrm{dx}}{\sqrt{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}}$

then express $\mathrm{px}+\mathrm{q} = \mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c})+\mathrm{m}$

,compare the coefficient of x and constant term , find the value of l and m.

Now the given integration can be written in the form of $\int \frac{(\mathrm{px}+\mathrm{q})\,\mathrm{dx}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}$

$$= \int \frac{\mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c})+\mathrm{m}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}\mathrm{dx} = \int \frac{\mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c})}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}} + \int \frac{\mathrm{m}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}\mathrm{dx}$$

Solve it by using the formula.

Ex: $\int \frac{(3x+2)dx}{x^2+4x+9}$

Ans: $\int \frac{(3x+2)dx}{x^2+4x+9}$

Let $3x+2 = l\frac{d}{dx}(x^2+4x+9)+m$

$$= l(2x+4)+m$$

$$= 2lx+4l+m$$

Compare the coefficient of x and constant term in both the sides

$$2l = 3 \Rightarrow l = \frac{3}{2} \quad and \;\; 4l+m = 2 \Rightarrow m = 2-4l = 2-4\,.\frac{3}{2} = -4$$

$So\;\; \int \frac{(3x+2)dx}{x^2+4x+9} = \int \frac{l(2x+4)+m}{x^2+4x+9}dx = \int \frac{l(2x+4)}{x^2+4x+9}dx + \int \frac{m}{x^2+4x+9}dx$

$= l\int \frac{(2x+4)}{x^2+4x+9}dx + m\int \frac{dx}{x^2+4x+9}$

Let $x^2 + 4x + 9 = t \Rightarrow (2x+4)dx = dt$

$= l\int \frac{dt}{t} + m\int \frac{dx}{x^2+2.x.2+2^2-2^2+9} = l\ ln|t| + m\int \frac{dx}{(x+2)^2+5}$

Let $x + 2 = z \Rightarrow dx = dz$

$= l\ ln|t| + m\int \frac{dz}{(z)^2+(\sqrt{5})^2} = \frac{3}{2}ln|x^2 + 4x + 9| - 4tan^{-1}\frac{z}{\sqrt{5}} + c$

$= \frac{3}{2}ln|x^2 + 4x + 9| - 4tan^{-1}\frac{x+2}{\sqrt{5}} + c$

Assignment:

Ex: $\int \frac{dx}{4+9x^2}$

Ex: $\int \frac{3x}{1+2x^4}$

Ex: $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}}dx$

Ex: $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}dx$

Ex: $\int \frac{2^x}{\sqrt{9+4^x}}dx$

Ex: $\int \frac{dx}{\sqrt{16x^2+25}}$

Ex: $\int \frac{1}{\sqrt{a^2+b^2x^2}}dx$

Ex: $\int \frac{x^2}{\sqrt{x^6-1}}dx$

Ex: $\int \frac{e^{-x}}{16+9e^{-2x}}dx$

Ex: $\int \frac{1}{e^x+e^{-x}}dx$

Ex: $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

Ex: $\int \frac{5x-2}{\sqrt{11-x^2}}dx$

Ex: $\int \frac{3x+5}{\sqrt{x^2-9}}dx$

Ex: $\int \frac{3x-2}{16+x^2}dx$

Ex: $\int \frac{x-2}{\sqrt{4+x^2}}dx$

Ex: $\int \frac{dx}{9x^2-12x+8}$

Ex: $\int \frac{dx}{\sqrt{2-4x+x^2}}$

Ex: $\int \frac{dx}{2x^2+x+3}$

Ex: $\int \frac{1}{\sqrt{7-6x-x^2}}dx$

Ex: $\int \frac{2x+3}{\sqrt{5-4x-x^2}}dx$

INTEGRATION BY PARTS

If u and v are two functions then integration of the product of u and v is defined as

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

Here the 1st function can be choosed by using a form **ILATE**.

I-Inverse function $(\sin^{-1}x, \cos^{-1}x, \tan^{-1}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

L-Logarithm function $(\log x, \log(x+1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

A-algebraic function $(x, x^2, (x+1)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

T-Trigonometric function $(\sin x, \cos x, \tan x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

E-Exponential function $(e^x, a^x, e^{x+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

Ex: $\int x\,e^x\,dx$ here u = x and v= e^x

Ans: $\int x\,e^x\,dx = x\int e^x\,dx - \int\left\{\frac{d(x)}{dx}\int e^x dx\right\}dx$

$= x\,e^x - \int 1\,e^x dx = xe^x - e^x + c$

Ex: $\int x\,sinx\,dx$

Ans: $\int x\,sinx\,dx = x\int sinx\,dx - \int\left\{\frac{d(x)}{dx}\int sinx\,dx\right\}dx$

$= x\,(-cosx)\ - \int 1\,(-cosx)\,dx = -x\,cosx + sinx + c$

Ex: $\int x\,cosx\,dx$= $x\int cosx\,dx - \int\left\{\frac{d(x)}{dx}\int cosx\,dx\right\}dx$

$= x\,(sinx)\ - \int 1\,(sinx)\,dx = x\,sinx + cosx + c$

Ex: $\int x\, e^{2x}\, dx$

Ans: $\int x\, e^{2x}\, dx = x \int e^{2x}\, dx - \int \left\{\frac{d(x)}{dx} \int e^{2x} dx\right\} dx$

$= x\, \frac{e^{2x}}{2} - \int 1\, \frac{e^{2x}}{2}\, dx = x\, \frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x} dx + c = x\, \frac{e^{2x}}{2} - \frac{1}{2}\, \frac{e^{2x}}{2} + c$

Ex: $\int x\, sin3x\, dx$

Ans: $\int x\, sin3x\, dx = x \int sin3x\, dx - \int \left\{\frac{d(x)}{dx} \int sin3x\, dx\right\} dx$

$= x\, \left(\frac{-cos3x}{3}\right) - \int 1\, \left(\frac{-cos3x}{3}\right)\, dx = -x \frac{cos3x}{3} + \frac{1}{3}\int cos3x\, dx + c$

$= -x \frac{cos3x}{3} + \frac{1}{3} \frac{sin3x}{3} + c$

Ex: $\int x\, sec^2x\, dx$

Ans: $\int x\, sec^2x\, dx = x \int sec^2x\, dx - \int \left\{\frac{d(x)}{dx} \int sec^2x\, dx\right\} dx$

$= x\, tanx - \int 1\, tanx\, dx = x\, tanx - ln|secx| + c$

Ex: $\int (x+1)\, e^x\, dx$

Ans: $\int (x+1)\, e^x\, dx = (x+1) \int e^x\, dx - \int \left\{\frac{d(x+1)}{dx} \int e^x dx\right\} dx$

$= (x+1)e^x - \int 1\, e^x dx = (x+1)e^x - e^x + c$

OR

$\int (x+1)\, e^x\, dx = \int x\, e^x dx + \int e^x dx$

Ex: $\int x\, tan^2x\, dx$

Ans: $\int x\, tan^2x\, dx = \int x\, (sec^2x - 1) dx$

$= \int x\, sec^2x\, dx - \int x\, dx$

$= x \int sec^2x\, dx - \int \left\{\frac{d(x)}{dx} \int sec^2x\, dx\right\} dx - \frac{x^2}{2}$

$= x\, tanx - \int 1\, tanx\, dx - \frac{x^2}{2} = x\, tanx - ln|secx| - \frac{x^2}{2} + c$

Ex: $\int x\, cos^2x\, dx$

Ans: $\int x\, cos^2x\, dx = \int x\, \left(\frac{1+cos2x}{2}\right) dx = \int \frac{x}{2} dx + \int \frac{x\, cos2x}{2}\, dx$

$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\,cos2x\,dx$

$= \frac{1}{2}\,\frac{x^2}{2} + \frac{1}{2}\left\{x\int cos2x\,dx - \int\left\{\int\frac{d(x)}{dx}\int cos2x\,dx\right\}dx\right\}$

$= \frac{x^2}{4} + \frac{1}{2}\left\{x\,\left(\frac{sin2x}{2}\right) - \int 1\,\left(\frac{sin2x}{2}\right)\,dx\right\}$

$= \frac{x^2}{4} + \frac{1}{2}\left\{x\,\left(\frac{sin2x}{2}\right) - \frac{1}{2}\int\,sin2x\,dx\right\}$

$= \frac{x^2}{4} + \frac{1}{2}\left\{x\,\left(\frac{sin2x}{2}\right) - \frac{1}{2}\left(\frac{-cos2x}{2}\right)\right\} + c$

Ex:$\int x\,sin3x\,cos2x\,dx$

Ans: $\int x\,sin3x\,cos2x\,dx = \frac{1}{2}\int x\,(2sin3x\,cos2x)\,dx$

$= \frac{1}{2}\int x\,(\sin(3x+2x) + \sin(3x-2x))\,dx$

$= \frac{1}{2}\int x\,(\sin 5x + \sin x)dx = \frac{1}{2}\int x\;\,sin5x\,dx + \frac{1}{2}\int x\sin x\,dx$

Ex:$\int lnx\,dx$

Ans: $\int lnx\,dx = lnx\int 1\,dx - \int\left\{\frac{d(lnx)}{dx}\int 1\,dx\right\}dx$

$= lnx\,x - \int\frac{1}{x}\,x\,dx = x\,lnx - \int dx = x\,lnx - x + c$

Ex:$\int x^5\,lnx\,dx$

Ans: $\int x^5\,lnx\,dx = lnx\int x^5\,dx - \int\left\{\frac{d(lnx)}{dx}\int x^5\,dx\right\}dx$

$= lnx\,\frac{x^6}{6} - \int\frac{1}{x}\,\frac{x^6}{6}\,dx = \frac{x^6}{6}lnx - \frac{1}{6}\int\,x^5dx = \frac{x^6}{6}lnx - \frac{1}{6}\frac{x^6}{6} + c$

$= \frac{x^6}{6}lnx - \frac{x^6}{36} + c$

Ex:$\int x\ln(1+x)\,dx$

Ans: $\int x\ln(1+x)\,dx = \ln(x+1)\int x\,dx - \left\{\frac{d(\ln(x+1))}{dx}\int x\,dx\right\}dx$

$= \ln(x+1)\ \frac{x^2}{2} - \int \frac{1}{x+1}\ \frac{x^2}{2}\, dx = x\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\, dx$

$= x\ln(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\, dx$

$= x\ln(x+1) - \frac{1}{2}\left\{\int \frac{x^2-1}{x+1}\, dx + \int \frac{1}{x+1}\, dx\right\}$

$= x\ln(x+1) - \frac{1}{2}\left\{\int \frac{(x+1)(x-1)}{x+1}\, dx + \int \frac{1}{x+1}\, dx\right\}$

$= x\ln(x+1) - \frac{1}{2}\int (x-1) dx - \frac{1}{2}\int \frac{1}{x+1}\, dx$

$= x\ln(x+1) - \frac{1}{2}\int x\, dx + \frac{1}{2}\int dx - \frac{1}{2}\int \frac{1}{x+1}\, dx$

$= x\ln(x+1) - \frac{1}{2}\ \frac{x^2}{2} + \frac{1}{2}\ x - \frac{1}{2} ln|x+1| + c$

IMP

Ex: $\int \ln(1+x^2)\, dx$

Ans: $\int \ln(1+x^2)\, dx = \ln(1+x^2)\int 1\, dx - \int \left\{\frac{d}{dx} ln(1+x^2)\int 1\, dx\right\} dx$

$= \ln(1+x^2)\ x - \int \frac{1}{1+x^2}\ 2x\ x\, dx = x\ ln(1+x^2) - 2\int \frac{x^2}{1+x^2} dx$

$= x\ ln(1+x^2) - 2\int \frac{x^2+1-1}{1+x^2} dx = x\ ln(1+x^2) - 2\int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right) dx$

$= x\ ln(1+x^2) - 2\int dx + 2\int \frac{1}{1+x^2} dx = x\ ln(1+x^2) - 2x + 2\ tan^{-1}x + c$

Ex: $\int sin^{-1}x\, dx$

Ans: $\int sin^{-1}x\, dx = sin^{-1}x\ \int 1\, dx - \int \left\{\frac{d}{dx} sin^{-1}x \int 1\, dx\right\} dx$

$= sin^{-1}x\ .x - \int \frac{1}{\sqrt{1-x^2}}\ x\, dx = x.sin^{-1}x - \int \frac{x\, dx}{\sqrt{1-x^2}}$

For 2nd term let $1 - x^2 = t^2 \Rightarrow -2x\, dx = 2t\, dt \Rightarrow x\, dx = -t\, dt$

$= x.sin^{-1}x - \int \frac{-t\,dt}{t} = x.sin^{-1}x + t + c = x.sin^{-1}x + \sqrt{1-x^2} + c$

Ex: $\int tan^{-1}x\,dx$

Ans: $\int tan^{-1}x\,dx = tan^{-1}x \int 1\,dx - \int \left\{\frac{d}{dx} tan^{-1}x \int 1\,dx\right\} dx$

$= tan^{-1}x\ .x - \int \frac{1}{1+x^2}\ x\,dx = x.tan^{-1}x - \int \frac{x\,dx}{1+x^2}$

For 2nd term let $1 + x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = dt/2$

$= x.tan^{-1}x - \int \frac{dt/2}{t} = x.tan^{-1}x + \frac{1}{2} ln|t| + c$

$= x.tan^{-1}x + \frac{1}{2} ln|1 + x^2| + c$

Ex: $\int x\ tan^{-1}x\,dx$

Ans: $\int x\ tan^{-1}x\,dx = tan^{-1}x \int x\,dx - \int \left\{\frac{d}{dx} tan^{-1}x \int x\,dx\right\} dx$

$= tan^{-1}x\ .\frac{x^2}{2} - \int \frac{1}{1+x^2}\ .\frac{x^2}{2}\ dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int \frac{x^2\,dx}{1+x^2}$

$= \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\left\{\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right\} dx$

$= \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2} dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\ x + \frac{1}{2} tan^{-1}x + c$

Assignment:

Ex: $\int x\ sinx\ cosx\,dx$

Ex: $\int x\ sin5x\ dx$

Ex: $\int xe^{bx} dx$

Ex: $\int xcos^2x\,dx$

Ex: $\int xcosnx\,dx$ Ex: $\int x\ lnx\,dx$

Ex: $\int \frac{lnx}{x^5}\ dx = \int x^{-5} lnx\,dx$

Ex: $\int x^n\ lnx\,dx$ Ex: $\int cos^{-1}x\,dx$

Ex: $\int (lnx)^2 dx$

Formula:

$$\int e^x(f(x)+f'(x)) = e^x f(x) + c$$

1. $\int e^x(tanx + lnsecx)dx$

2. $\int \frac{xe^x}{(1+x)^2}dx$

3. $\int e^x \frac{1-sinx}{1-cosx}dx$

4. $\int e^x \frac{2+sin2x}{1+cos2x}\,dx$

Formulas:1

$$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + c$$

Proof: $Let\ I = \int \sqrt{a^2-x^2}dx = \sqrt{a^2-x^2}\int dx - \int\left(\frac{d\sqrt{a^2-x^2}}{dx}\int dx\right)dx$

$$= x\sqrt{a^2-x^2} - \int \frac{1}{2\sqrt{a^2-x^2}}(-2x)x\,dx = x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2-x^2}}dx$$

$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}dx$

$\Rightarrow I + I = x\sqrt{a^2-x^2} + a^2 sin^{-1}\frac{x}{a} + c_1$ $\quad [\because \int \sqrt{a^2-x^2}\,dx = I]$

$\Rightarrow 2I = x\sqrt{a^2-x^2} + a^2 sin^{-1}\frac{x}{a} + c_1 \Rightarrow I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + c$

$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + c$

2.

$$\int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}ln\left|x+\sqrt{a^2+x^2}\right| + c$$

Proof: $Let\ I = \int \sqrt{a^2+x^2}dx = \sqrt{a^2+x^2}\int dx - \int\left(\frac{d\sqrt{a^2+x^2}}{dx}\int dx\right)dx$

$$= x\sqrt{a^2+x^2} - \int \frac{1}{2\sqrt{a^2+x^2}}(2x)x\,dx = x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}\,dx = x\sqrt{a^2+x^2} - \int \frac{a^2+x^2}{\sqrt{a^2+x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2+x^2}}dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2+x^2}}dx$$

$$\Rightarrow I + I = x\sqrt{a^2+x^2} + a^2 ln\left|x+\sqrt{a^2+x^2}\right| + c_1 \qquad [\because \int \sqrt{a^2+x^2}\,dx = I]$$

$$\Rightarrow 2I = x\sqrt{a^2+x^2} + a^2 ln\left|x+\sqrt{a^2+x^2}\right| + c_1 \Rightarrow I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} ln\left|x+\sqrt{a^2+x^2}\right| + c$$

$$\int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} ln\left|x+\sqrt{a^2+x^2}\right| + c$$

3.

$$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} ln\left|x+\sqrt{x^2-a^2}\right| + c$$

Proof: $Let\ I = \int \sqrt{x^2-a^2}dx = \sqrt{x^2-a^2}\int dx - \int\left(\frac{d\sqrt{x^2-a^2}}{dx}\int dx\right)dx$

$$= x\sqrt{x^2-a^2} - \int \frac{1}{2\sqrt{x^2-a^2}}(2x)x\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}dx$$

$$\Rightarrow I + I = x\sqrt{x^2-a^2} - a^2 ln\left|x+\sqrt{x^2-a^2}\right| + c_1 \qquad [\because \int \sqrt{x^2-a^2}\,dx = I]$$

$$\Rightarrow 2I = x\sqrt{x^2-a^2} - a^2 ln\left|x+\sqrt{x^2-a^2}\right| + c_1 \Rightarrow I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} ln\left|x+\sqrt{x^2-a^2}\right| + c$$

4.

$$\int e^{ax}cosbx\,dx = \frac{e^{ax}}{a^2+b^2}[acosbx + bsinbx] + c$$

Proof:Let I=$\int e^{ax} cosbxdx = cosbx \int e^{ax} dx - \left(\int \left(\frac{dcosbx}{dx} \int e^{ax} dx\right) dx\right)$

$= cosbx \frac{e^{ax}}{a} + \int sinbx . b \frac{e^{ax}}{a} dx = \frac{e^{ax}}{a} cosbx + \frac{b}{a} \int e^{ax} sinbx\, dx$

$= \frac{e^{ax}}{a} cosbx + \frac{b}{a}\left[\frac{e^{ax}}{a} sinbx - \left(\int \left(\frac{dsinbx}{dx} \int e^{ax} dx\right) dx\right)\right]$

$= \frac{e^{ax}}{a} cosbx + \frac{b}{a^2} e^{ax} sinbx - \frac{b^2}{a^2} \int e^{ax} cosbxdx = \frac{e^{ax}}{a} cosbx + \frac{b}{a^2} e^{ax} sinbx - \frac{b^2}{a^2} I$

$I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a} cosbx + \frac{b}{a^2} e^{ax} sinbx$

$\Rightarrow \frac{a^2+b^2}{a^2} I = \frac{a e^{ax} cosbx + b e^{ax} sinbx}{a^2} + c$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(acosbx + bsinbx)$+c

$\int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}[acosbx + bsinbx] + c$

5.

$$\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2 + b^2}[asinbx - bcosbx] + c$$

Proof:Let I=$\int e^{ax} sinbxdx = sinbx \int e^{ax} dx - \left(\int \left(\frac{dsinbx}{dx} \int e^{ax} dx\right) dx\right)$

$= sinbx \frac{e^{ax}}{a} - \int cosbx . b \frac{e^{ax}}{a} dx = \frac{e^{ax}}{a} sinbx - \frac{b}{a} \int e^{ax} cosbx\, dx$

$= \frac{e^{ax}}{a} sinbx - \frac{b}{a}\left[\frac{e^{ax}}{a} cosbx - \left(\int \left(\frac{dcosbx}{dx} \int e^{ax} dx\right) dx\right)\right]$

$= \frac{e^{ax}}{a} sinbx - \frac{b}{a^2} e^{ax} cosbx - \frac{b^2}{a^2} \int e^{ax} sinbxdx = \frac{e^{ax}}{a} sinbx - \frac{b}{a^2} e^{ax} cosbx - \frac{b^2}{a^2} I$

$I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a} sinbx - \frac{b}{a^2} e^{ax} cosbx$

$\Rightarrow \frac{a^2+b^2}{a^2} I = \frac{a e^{ax} sinbx - b e^{ax} cosbx}{a^2} + c$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx)$+c

$\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}[asinbx - bcosbx] + c$

Assignment:

1. $\int \frac{x^2}{1-x^6} dx$
2. $\int \frac{dx}{9x^2-7}$
3. $\int \frac{dx}{x^2+2x+6}$
4. $\int \frac{dx}{9x^2-12x+8}$
5. $\int \frac{dx}{1+x-x^2}$
6. $\int \frac{x}{x^4+x^2+1} dx$
7. $\int \frac{x}{x^2+x+1} dx$
8. $\int \frac{2x-3}{x^2+3x-18} dx$
9. $\int \frac{dx}{\sqrt{2-4x+x^2}}$
10. $\int \frac{dx}{\sqrt{8+3x-x^2}}$
11. $\int \frac{2x+1}{\sqrt{x^2+2x-1}} dx$
12. $\int \frac{x+1}{\sqrt{4+5x-x^2}} dx$
13. $\int e^{3x} sin4x\ dx$
14. $\int e^{-3x} sin2x\ dx$
15. $\int e^{-x} cos4x\ dx$
16. $\int e^{-x} cosx\ dx$
17. $\int e^{ax} \sin(bx+c) dx$

DEFINITE INTEGRATION

Let f(x) be a continued function defined in an interval [a,b],so definite integral of f(x) over [a,b] is defined as $\int_a^b f(x)dx = [F(x)+c]_a^b = F(a) - F(b)$

$$Here\ x = a\ is\ the\ lower\ limit\ and\ x = b\ is\ the\ upper\ limit$$

In definite integration constant c is deleted.

Ex:$Evaluate\ \int_2^3 x^2\ dx$

Ans: $\int_2^3 x^2\ dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

Ex:$Evaluate\ \int_0^{\frac{\pi}{4}} \frac{1}{1+x^2} dx$

Ans:$\int_0^{\frac{\pi}{4}} \frac{1}{1+x^2} dx = [tan^{-1}x]_0^{\frac{\pi}{4}} = tan^{-1}\frac{\pi}{4} - tan^{-1}0 = 1 - 0 = 1$

Ex: $Evaluate\ \int_0^{\frac{\pi}{2}} xsinx\,dx$

Ans: $\int xsinx\,dx = x\int sinx\,dx - \int(\frac{d(x)}{dx}\int sinxdx)dx = -xcosx - \int 1(-cosx)dx$

$= -xcosx + sinx$

Now $\int_0^{\frac{\pi}{2}} xsinx\,dx = [-xcosx + sinx]_0^{\frac{\pi}{2}} = (-\frac{\pi}{2}cos\frac{\pi}{2} + sin\frac{\pi}{2}) - (0 + sin0) = 1$

Ex: $Evaluate\ \int_0^{\pi} tan^2x\,dx$

Ans: $\int_0^{\pi} tan^2x\,dx = \int_0^{\pi}(sec^2x - 1)dx = \int_0^{\pi} sec^2xdx - \int_0^{\pi} dx = [tanx]_0^{\pi} - [x]_0^{\pi} = -\pi$

Definite integration by substitution:

Ex: $Evaluate\ \int_1^2 xe^{x^2}dx$

Ans: $\int_1^2 xe^{x^2}dx$ $\quad let\ x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$

$limit\ change\ \begin{cases} x = 1, t = 1 \\ x = 2, t = 4 \end{cases}$ $\quad for\ finding\ the\ value\ of\ t\ put\ the\ value\ of\ x\ in\ x^2 = t.$

$=\int_1^4 e^t\frac{dt}{2} = \frac{1}{2}[e^t]_1^4 = \frac{1}{2}(e^4 - e^1)$

Ex: $Evaluate\ \int_0^{\frac{\pi}{2}} sin^3x\cos dx$

Ans: $\int_0^{\frac{\pi}{2}} sin^3x\cos dx$ $\quad let\ sinx = t \Rightarrow cosx\,dx = dt$

$\begin{cases} x = 0, t = 0 \\ x = \frac{\pi}{2}, t = 1 \end{cases}$.

$=\int_0^1 t^3dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

Properties:

1. $\int_a^b f(x)dx = \int_a^b f(y)dy$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

4. $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if\ f(x) is\ an\ even\ function \\ 0, If\ f(x) is\ an\ odd\ function \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if\ f(2a-x) = f(x) \\ 0, if\ f(2a-x) = -f(x) \end{cases}$

Reference Books: Elements of Mathematics Vol. - 2 (Odisha State Bureau of Text Book preparation & Production)