KIIT POLYTECHNIC

LECTURE NOTES

ON

ENGG. MATH -I

PART-2

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CONTENTS

CHAPTER-3
Co-ordinate Geometry (2D)

Co-ordinate : It represents the position of a point on a plane by an ordered pair of real numbers.

Co-ordinate Geometry: Coordinate geometry (or analytic geometry) is defined as the study of geometry using the coordinate points.

Why do we Need Coordinate Geometry?

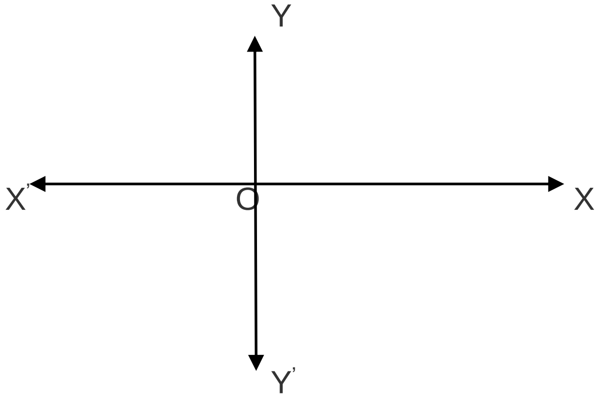
Coordinate geometry has various applications in real life. Some of the areas where coordinate geometry is an integral part include.

- In digital devices like computers, mobile phones, etc. to locate the position of cursor or finger.
- In aviation to determine the position and location of airplanes accurately.
- In maps and in navigation (GPS).
- To map geographical locations using latitudes and longitudes.

Rectangular co-ordinate system: If the horizontal line $XOX^{'}$ *and the vertical line* $YOY^{'}$ are perpendicular to each other then it is called rectangular co-ordinate system.

Here XOX' *is called* $X-$ *axis and* YOY' *is called* $Y-$ *axis and* they are intersect at O(called origin).OX and OY are the +ve direction of X-axis and Y-axis but OX' *and* OY' are –ve direction of X-axis and Y-axis.

Figure:

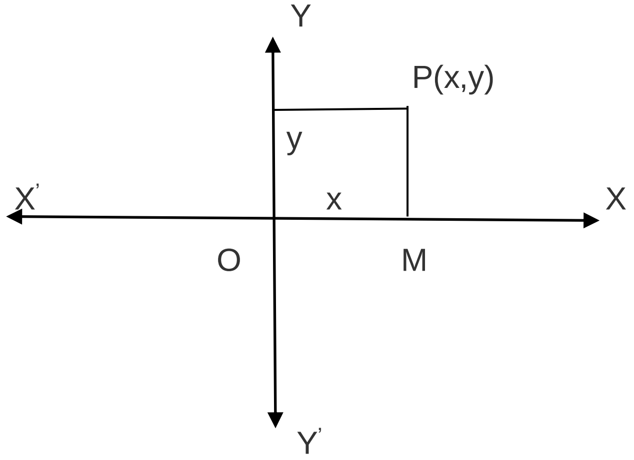


Cartesian co-ordinate:

Let P be any point on the plane , Draw a perpendicular from P on X-axis,which meets at M. Let OM=x and MP=y,so the ordered pair (x,y) represents the co-ordinate of the point P.Which is written as P(x,y).(read as P be a point having co-ordinate (x,y))

Here x=x-coordinate or abscissa and y=y-coordinate or ordinate

Figure:



Distance Formula:

Theorem: The distance between any two pints $P(x_1, y_1) and\ Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note:1.The distance is always positive.

2.The distance from origin to any point $P(x, y)$ is given by $OP = \sqrt{x^2 + y^2}$

3.If a point $P(x, y)$ lies on x-axis then y=0 ,so the co-ordinate of any point on x-axis is P(x,0) and the equation of x-axis is y=0.

4. If a point $P(x, y)$ lies on y-axis then x=0 ,so the co-ordinate of any point on y-axis is P(0,y) and the equation of y-axis is x=0.

5.The distance from any point P(x,y) on X-axis and y-axis is

$$PL = |y|\ and\ PM = |x|$$

Area of a triangle:

Theorem: The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)\ and\ (x_3, y_3)$ is given by

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Or $\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

Collinear of Three Points:

Three points are said to be collinear if they lie in one straight line.

Condition of Collinearity: Three points will be collinear if area of the triangle is zero.

Or AB+BC=AC

Division Formula:

Formula for Internally Division:

Theorem:The coordinates of the point P(x,y) which divides the line segment joining $A(x_1, y_1) and\ B(x_2, y_2)\ in\ the\ ratio\ l{:}m\ internally\ is\ given\ by$

$$x = \frac{lx_2+mx_1}{l+m}, y = x = \frac{ly_2+my_1}{l+m}$$

Formula for Externally Division:

Theorem:The coordinates of the point P(x,y) which divides the line segment joining $A(x_1, y_1) and\ B(x_2, y_2)\ in\ the\ ratio\ l{:}m\ externally\ is\ given\ by$

$$x = \frac{lx_2-mx_1}{l-m}, y = x = \frac{ly_2-my_1}{l-m}$$

Mid point Formula:

Theorem:The coordinates of the mid point P(x,y) on the line joining $A(x_1, y_1) and$

$B(x_2, y_2)\ in\ the\ ratio\ l{:}m\ externally\ is\ given\ by\quad x = \frac{x_2+x_1}{2}, y = x = \frac{y_2+y_1}{2}$

Centroid of a triangle:The point at which all the medians of a triangle are intersect is known as centroid of a triangle.

Note: Centroid of a tiangle always divides the median in the ratio 2:1.

Centroid Formula:

Theorem:The co-ordinates of the centroid of a triangle ABC with vertices $A(x_1, y_1)$,

$B(x_2, y_2)\ and\ C(x_3, y_3)\ is\ given\ by\ x = \frac{x_2+x_1+x_3}{3}, y = x = \frac{y_2+y_1+y_3}{3}$

Incentre: The point at which all the angle bisectors of a triangle are intersect is known as incentre of a triangle.

Incentre Formula:

Theorem: The co-ordinates of the Incentre of a triangle ABC with vertices $A(x_1, y_1)$,

$B(x_2, y_2)\ and\ C(x_3, y_3)\ is\ given\ by\ x = \frac{ax_2+ax_1+cx_3}{a+b+c}, y = \frac{by_2+ay_1+cy_3}{a+b+c}$

Definition: Slope of a Line

A line in a two dimensional plane forms two types of angles with the x-axis, which are supplementary (i.e. sum of the angles is 180^0).

Referring the figures 1 and 2, Can you say which type of staircase do you prefer to climb a roof or hill? Obviously, one must prefer 1st one. Can you guess why? In the 1st case the steps are less steep than 2nd one.

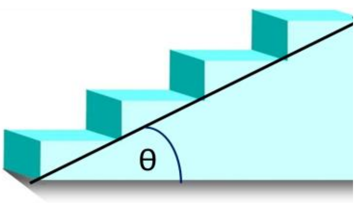


Figure 1

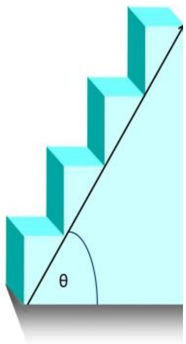


Figure 2

Mathematically, the angle θ made by the line along the steps with horizontal line (positive direction of x-axis) is called as inclination of the line. For inclination, the angle θ is always measured in positive direction of x-axis only. In the figure 1 the inclination is less than that in figure 2.

Formal Definition of Inclination: The angle (say) θ made by a line with positive direction of x-axis, measured anti clockwise is called the inclination of the line. Clearly, $0° \le \theta \le 180°$

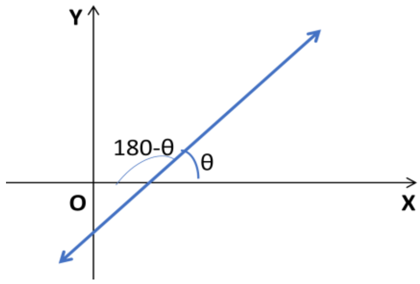


Figure 3

Points to remember: Lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°. The inclination may be either acute or obtuse.

Definition: If θ is the inclination of a line L, then the slope or gradient of the line L is defined as tanθ. The slope of a line with inclination 90° is not defined. The slope of a line is also written as m. Thus, $m = \tan\theta$, $\theta \neq 90°$ .It may be noted that the slope of x-axis is zero and slope of y-axis is not defined.

For instance, the slope of a line with inclination 30^0 will be $\tan 30^0 = \frac{1}{\sqrt{3}}$. Conversely, the inclination of a line with slope $\frac{1}{\sqrt{3}}$ will be 30^0.

Now answer the following:

Find the slope of the following lines whose inclinations are:

a) 45^0 b) 60^0 c) 135^0 d) 150^0 e) 120^0

Slope of a line when coordinates of any two points on the line are given

A line is completely determined when two points are given on it. Let us proceed to find the slope of a line in terms of the coordinates of two points on the line. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on non-vertical line L whose inclination is θ. Clearly, $x_1 \neq x_2$, otherwise the line will be perpendicular to x-axis and its slope will not be defined. Draw perpendiculars AP, BQ to x-axis and AC to BQ as shown in figure 4.

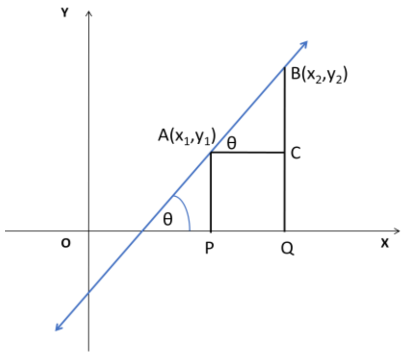


Figure 4

In the triangle ABC, $\tan\theta = \frac{BC}{AC} = \frac{BC}{PQ}$

$$= \frac{BQ - CQ}{OQ - OP} = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus, if (x_1, y_1) and (x_2, y_2) are coordinates of any two points on a line, then its slope is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{------------- (1.1)}$$

Let us try to answer the following questions.

1. Find the slope of a line which passes through points (3, 2) and (-1, 5).

Methods to solve:

What are x_1, y_1 and x_2, y_2?

Use these values in the above formula.

What is the answer? Is it $\frac{-3}{4}$.

2. Determine x so that 2 is the slope of the line through (2, 5) and (x , 3).

Methods to solve:

What is the value of given slope?

Find the slope of the line using the formula.

Then equate them to get x.

3. How to write the solution of the following question:

Que. Find the slope of the line segment joining the points (-3, 7) and (2, 9).

Solution:

Here, $x_1=-3$, $y_1=7$, $x_2=2$, $y_2=9$.

$$\text{Slope}= m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-7}{2-(-3)} = \frac{2}{5} \qquad \text{(Answer)}$$

Try the following:

Find the slope of the lines:

(a) Passing through the points (-3, 2) and (1, -4),

(b) Passing through the points (3, –1) and (4, – 2),

(c) Passing through the points (3, – 7) and (5, 9)

Angle between two lines:

When we think about more than one line lying in a plane, then we find that these lines are either intersecting or parallel. Here we will find the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. Let θ_1 and θ_2 be the inclinations of lines L_1 and L_2 respectively. Then $m_1=\tan\theta_1$ and $m_2= \tan\theta_2$.

From figure 5, $\theta + \theta_1 = \theta_2 \Rightarrow \theta = \theta_2 - \theta_1$

$$\Rightarrow tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow tan\theta = \frac{tan\theta_2 - tan\theta_1}{1 + tan\theta_2 tan\theta_1}$$

$$\Rightarrow tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

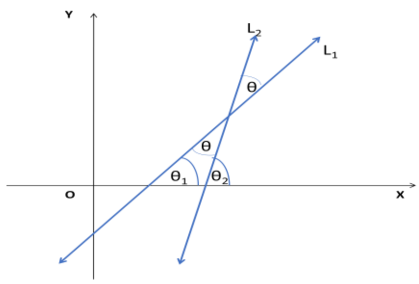


Figure 5

If θ is acute, then $tan\theta > 0$, and

If θ is obtuse, then $tan\theta < 0$.

Hence,

$$tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right) \qquad \text{--------- (1.2)}$$

In other words, one can say to find the acute angle between two lines with given slopes use:

$tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|$ -------- (1.3)

Let us know what happens when $\theta = 0$ or $\theta = 90°$.

If $\theta = 0$, then formula (1.3) becomes $tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right| \Rightarrow tan0 = \left|\frac{m_2-m_1}{1+m_1m_2}\right| \Rightarrow \left|\frac{m_2-m_1}{1+m_1m_2}\right| = 0$

$\Rightarrow \frac{m_2-m_1}{1+m_1m_2} = 0 \Rightarrow m_2 = m_1$, is called as parallel condition for two lines in a plane.

If $\theta = 90°$, then $cot90° = 0 \Rightarrow \frac{1}{tan90°} = 0 \Rightarrow \left|\frac{1+m_1m_2}{m_2-m_1}\right| = 0 \Rightarrow 1 + m_1m_2 = 0 \Rightarrow m_1m_2 = -1$, is called as perpendicular conditions for two lines in a plane.

1. Parallel lines have equal slope.
2. For two perpendicular lines the product of their slopes is -1.

Example:

If the slopes of two lines are given as $\sqrt{3}$ and $\frac{2}{\sqrt{3}}$, then find the acute angle between them.

Solution:

What is m_1 and m_2?
Find m_1 and m_2?
Use these values in formula (1.3). Is it 45^o?

Example:

If the angle between two lines is π/4 and slope of one of the lines is 1/2, find the slope of the other line.

Solution:

What is θ?
What is m_1 and m_2?
Let m_1=1/2.
Use formula (1.3) to get m_2. Is it -1/3 or 3?

Example:

If the line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (*x*, 24) find the value of *x*.

Solution:

Slope of the line through the points (– 2, 6) and (4, 8) is $\frac{8-6}{4-(-2)} = \frac{1}{3}$. Take this as m_1.

Slope of the line through the points (8, 12) and (*x*, 24) is $\frac{24-12}{x-8} = \frac{12}{x-8}$. Take this as m_2.

Since, the lines are perpendicular therefore, $m_1m_2 = -1$.

Use the values of m_1 and m_2 in this equation to get x.

Collinear points

Previously, we know that slopes of two parallel lines are equal. If two lines are parallel and passing through a common point, then two lines will coincide. Hence, if A, B and C are three points lying in the XY-plane, then they will be collinear if and only if slope of AB = slope of BC.

Example:

Find the value of *x* for which the points (*x,* – 1), (2*,* 1) and (4, 5) are collinear.

Solution:

Assume A as (*x,* – 1), B as (2*,* 1), and C as (4, 5).

Slope of AB is 2/(2-x), and slope of BC is 4/2=2.

Since, A, B, and C are collinear therefore slope of AB=slope of BC.

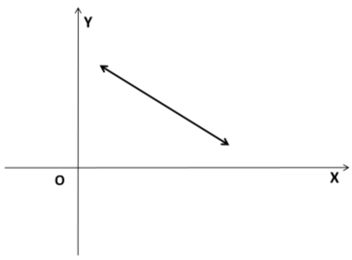
$\Rightarrow \frac{2}{2-x} = 2 \Rightarrow x = 2$

Exercise-1

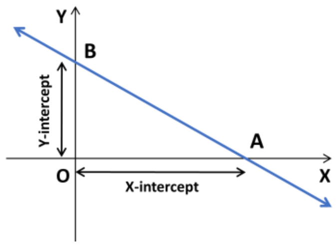
1. Find the mid-point of the line segment joining the points A (1, –2) and B (-9, 0).
2. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points A (0, – 4) and B (8, 6).
3. Find the value of *x* for which the points (*x,* – 1), (2,1) and (4, 7) are collinear.
4. Find the angle between the lines $\overline{AB}$ and $\overline{CD}$, where, A, B, C and D are (0, 1), (-2, 3), (2, -1), and (3, 4) respectively.

Intercepts of a Line

Now let us take a line L in XY-plane. Observe, what happens if the line is extended on either sides (figure 6).You can see the line will cut the x-axis and y-axis exactly at two different points A and B (figure 7).



(Figure 6)



(Figure 7)

Now we can see in figure 7 that the portion AB which is called as the portion of a line intercepted by the coordinate axes. We call the portion OA as x-intercept and OB as y-intercept of the line. In other words, X-intercept of the line is the portion of the line cut by the line with x-axis from the origin. Similarly, Y-intercept of the line is the portion of the line cut by the line with y-axis from the origin. Henceforth, you can keep in your mind x-intercept as 'a' and y-intercept as 'b'. It may be either positive or negative (refer figure 7).

> If a line has x-intercept 'a' then the line cuts x-axis at (a, 0).
> If a line has y-intercept 'b' then the line cuts y-axis at (0, b).

Various Forms of the Equation of a Line

As discussed earlier, the line is a locus and every line in a plane can be obtained by joining infinitely many points on it. So while drawing a line in a plane it will satisfy some condition like; how much inclination is? What is the x-intercept or y-intercept? Passing through some points and (or) Parallel/Perpendicular to other lines. Let P(x, y) be a variable point present on the line. The relationship between x- and y-coordinates of P satisfying some of the geometrical conditions is called as an equation:

Example:

The equation y=3x-1 is an equation of a line whose slope is 3 and

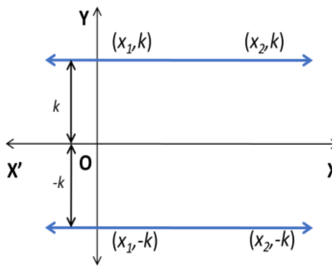


Figure 8

Y-intercept is -1.

Case-1 *Horizontal and vertical lines*

If a horizontal line L is at a distance *'k'* from *x*-axis, then the ordinate of every point lying on the line is either k or $-k$ (Fig 8). Therefore, equation of the line L is either $y = k$ or $y = -k$. Choice of sign will depend upon the position of the line lying above or below the *y*-axis accordingly.

Similarly, the equation of a vertical line at a distance *'h'* from the *y*-axis is either $x = h$ or $x = -h$ (Fig 9).

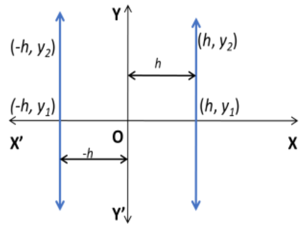


Figure 9

Example: Find the equation of the line passing through (-2, 5) and parallel to x-axis.

Solution:

Find the y-coordinate of the given point.

Put in the formula y=k. (Ans. y=5)

Example: Find the equation of the line passing through (-4, -7) and parallel to y-axis.

Solution:

Find the x-coordinate of the given point.

Put in the formula x=h. (Ans. x=-4)

Case-2 Slope-Point form

We can get equation of a straight line, provided, we are given any two characteristic data related to the line. Suppose we have been given the slope of the line as 'm' and also it is given that point (x_1, y_1) is on the line. If we think in question form, we may be asked like:

What is the equation of the straight line, if the straight line has a slope of 0.5 and it passes through (2, 3)?

We shall use a formula commonly known as slope point formula.

Formula:

The equation of a line passing through the fixed point A (x_0, y_0) and having slope as m is given by:

$$y - y_0 = m(x - x_0)$$

Proof:

To find the equation of the line let us take a variable point P(x,y) on the line so that we can use the given conditions to find a relation between x- and y-coordinates of P(Figure 10).

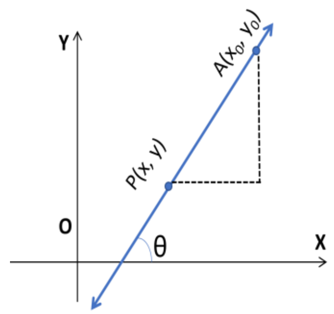


Figure 10

Hence, slope = $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_0}{x - x_0}$

$\Rightarrow y - y_0 = m(x - x_0)$ (Proved)

When you are asked to find the equation of a line with conditions stated, what is m?

What are values of (x_0, y_0)

Use the formula!

y-3=0.5(x-2)

It is always advised to simplify.

Example: Find the equation of the line passing through the point (2,-3) and slope as -1/2.
Method of solution:
Here find slope=m=?
What is x_0, y_0?
Use point slope formula. (Ans. x+2y+4=0)
Example: Find the equation of the line passing through the point (1,-5) and with inclination 60^o.
Method of solution:
Find slope=m=tan θ=tan60=?
What is x_0, y_0?
Use point slope formula. (Ans. √3x-y-√3-5=0)
Case- 3 Two-point form
Think of a line passing through two points. Can you find the slope of the line?
Yes, we have a formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Can you now find the equation of a line passing through two given points (-1, 4) and (3, 5)?
So we shall use a formula commonly known as *two-point formula*.

Formula: The equation of the line passing through the points (x1, y1) and (x2, y2) is:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Proof:

Let the line L passes through two given points A(x1, y1) and B(x2, y2). Let m=slope of the line L.

Then $m = \frac{y_2-y_1}{x_2-x_1}$

Using case-2 the equation of the line will be: $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$ (proved)

Now to find the equation of the line with conditions given above we will use this formula and hence the equation will be:

$$y - 5 = \frac{5-4}{3-(-1)}(x - 3)$$

$$\Rightarrow y - 5 = \frac{1}{4}(x - 3)$$ (Answer)

Example: Find the equation of the line passing through the points (-9, 2) and (3, 5).
Method of solution:
What are the values of x_1, y_1, x_2, and y_2?

Use Two-point formula to get the answer. (Answer: x-6y+27=0)

Example: Find the equation of the line passing through the points (1,-3) and (2,-4).

Solution:

Here x_1=1, y_1=-3, x_2=2, and y_2=-4.

Slope= $m = \frac{y_2-y_1}{x_2-x_1} = \frac{-4+3}{2-1} = -1$

Equation of the line will be: $y - y_1 = m(x - x_1)$

$\Rightarrow y - (-3) = -1(x - 1) \Rightarrow y + 3 = -x + 1 \Rightarrow x + y + 2 = 0$ (Answer)

Case-4 *Slope-Intercept form*

Now let us attempt to find the equation of a line whose slope (m) is given and y-intercept(c) is also given. We shall be using a formula known as *Slope-Intercept* formula.

Formula: The equation of the line with slope m and y-intercept as c will be:

y=mx+c

Proof:

To find the equation of the line, take an arbitrary point P(x, y) on the line L. As the line has y-intercept c so the line must pass through the point (0, c). Slope=m (given). We can apply the point-slope formula:

Hence, $y - c = m(x - 0)$

$\Rightarrow y - c = mx$

$\Rightarrow y = mx + c$

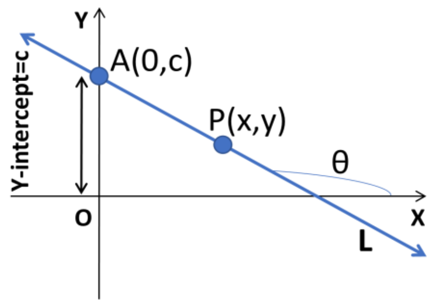


Figure 11

Example:

Find the equation of the line with slope 2/3 and y-intercept as -4.

Solution:

What is m?

What is c?

Use slope-intercept formula! (Answer: 2x-3y-12=0)

Case-5 *Intercept form*

Now can you find the equation of a line subject to the conditions?

i) If a line has x-intercept 6 and y-intercept 5.

OR,

ii) If a line cuts both the axes at given points (6, 0) and (0, 5).

Both these questions will expect same answer.

Formula:

The equation of the line with x-intercept a and y-intercept b will be:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof:

As the x-intercept of the line is a and y-intercept is b

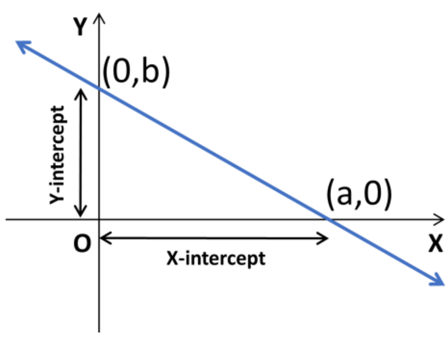


Figure 12

so, the line must cut the x-axis at (a, 0) and y-axis at (0, b) (figure 12).
To find the equation we can use two-point formula.
Hence,
Here x_1=a, y_1=0, x_2=0, y_2=b.
Equation of the line will be:

$$y - 0 = \frac{b-0}{0-a}(x-a) = -\frac{b}{a}(x-a)$$

$$\Rightarrow ay = -bx + ab$$

$$\Rightarrow bx + ay = ab$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$ (Dividing 'ab' on both sides) (proved)

In above question, a=6 and b=5.
Using the values in this formula we get, $\frac{x}{6} + \frac{y}{5} = 1$.

Example:
Find the equation of the line whose x- and y-intercepts are 2 and -5 respectively.

Method of solution:
What are the values of a, and b?
Use Intercept form? (Answer: 5x-2y-10=0)

Case-6 *Normal form*
What is the meaning of normal?
Think of two perpendicular lines. Then these two lines will be called as normal to each other.
Let us try to find the equation of a line whose distance from origin is given as p and the normal to the line makes an angle α with positive direction of x-axis.
You can be asked this in question form as:
Find the equation of the line which is at a distance of p from origin and the normal to the line makes an angle α with positive direction of x-axis.
Now we will use a formula called as Normal form.

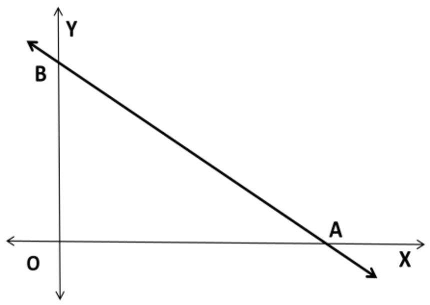


Figure 13a

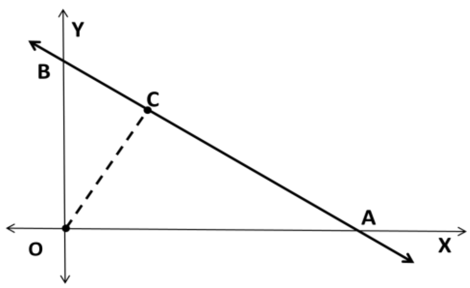


Figure 13b

Refer figure 13a, in which $\overleftrightarrow{AB}$ is any straight line. Can you find the length of the perpendicular drawn from the origin on the straight line $\overleftrightarrow{AB}$. It will be OC (figure 13b). Here OC is called as normal to the line $\overleftrightarrow{AB}$.

Formula:

If a line is at a distance of 'p' from origin and α is the angle made by the perpendicular drawn from the origin to the line (called as normal) with positive direction of x-axis. Then the equation of the line will be:

$$x\cos\alpha + y\sin\alpha = p$$

Proof:

Referring the figure 14 the normal to the line is the line segment OC. Let A and B be the points where the line meets the coordinate axes respectively. To find the equation of the line we need the values of x- and y-intercepts so that we can put these values in intercept form. In figure 14, OA= x-intercept and OB= y- intercept. The line segment OC is called as normal to the line.

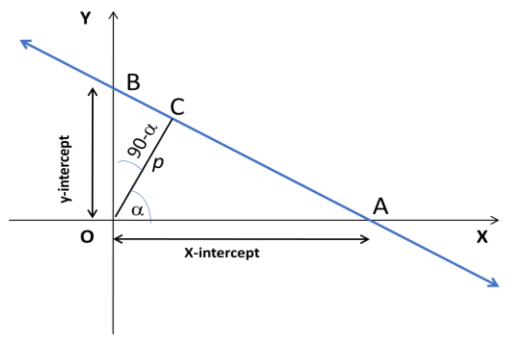


Figure 14

From the right-angle triangle OCA,

$$\cos\alpha = \frac{b}{h} = \frac{OC}{OA} = \frac{p}{OA}$$

$$\Rightarrow OA = \frac{p}{\cos\alpha}$$

Similarly from the right-angle triangle OCB

$$\cos(90-\alpha) = \frac{b}{h} = \frac{OC}{OB} = \frac{p}{OB}$$

$$\Rightarrow OB = \frac{p}{\cos(90-\alpha)}$$

$$\Rightarrow OB = \frac{p}{\sin\alpha}$$

Now using the intercept formula:

The equation of the line will be: $\frac{x}{OA} + \frac{y}{OB} = 1$

$$\Rightarrow \frac{x}{\frac{p}{\cos\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

$$\Rightarrow x\frac{\cos\alpha}{p} + y\frac{\sin\alpha}{p} = 1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p \qquad \text{(proved)}$$

Example: Find the equation of the line whose perpendicular distance from origin is 3 and the normal makes an angle 45^0 with positive direction of x-axis.

Method of solution:

What is p?

What is α?

Use these values in normal form! (Answer: $x + y = 3\sqrt{2}$)

Case-7 *General form*

Can you find a common form of all the equations discussed above?

Yes, all the equations of straight lines discussed in the above cases can be put into a form as:

$$ax + by + c = 0$$

called as general form.

Notes:

1. The slope of the line $ax + by + c = 0$ is $-\frac{a}{b}$.
2. The x-intercept of the line $ax + by + c = 0$ is $-\frac{c}{a}$.
3. The y-intercept of the line $ax + by + c = 0$ is $-\frac{c}{b}$.
4. The normal form of the line $ax + by + c = 0$ is $\frac{a}{\pm\sqrt{a^2+b^2}}x + \frac{b}{\pm\sqrt{a^2+b^2}}y = \frac{-c}{\pm\sqrt{a^2+b^2}}$ where the sign is chosen in order to make the R.H.S positive i.e. if c<0 then choose +ve sign on both sides and if c>0 then choose –ve sign on both sides.
5. From the note 4, the perpendicular distance of the line $ax + by + c = 0$ from the origin is $\frac{|c|}{\quad}$.

Example: Find the equation of the line passing through (-3, 2) and parallel to the line 3x-5y=12.

Method of solution:

Here, one can use point-slope formula.

What are x_0, y_0?

What is the relation between the slopes of two parallel lines? (i.e. Are they equal or different)

Hence, find the slope of the required line and assign this value to m.

Now use point-slope formula! (Answer; 3x -5y+19=0)

Example: Find the equation of the line passing through (-5, 7) and perpendicular to the line 2x-5y=9.

Method of solution:

Here, one can use point-slope formula.

What are x_0, y_0?

What is the relation between the slopes of two perpendicular lines? (i.e. Are they equal or product of their slopes is -1)

Hence, find the slope of the required line and assign this value to m.

Now use point-slope formula! (Answer; 5x+2y+11=0)

Example: Find the normal form of the equation of the line 2x-3y-5=0.

Method of solution:

What are a, b, and c?

Use these values in the normal-form and choose the appropriate sign(+ or -).

(Answer: $\frac{2}{\sqrt{13}}x - \frac{3}{\sqrt{13}}y = \frac{5}{\sqrt{13}}$)

Point of intersection of two lines

The point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ can be obtained by solving these two equations for x and y.

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$$\Rightarrow \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\frac{x}{\underset{12-21}{bc}} = \frac{y}{\underset{12-21}{ca}} = \frac{1}{\underset{12-21}{ab}}$$

Notes:

1. Two lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
2. Two lines will be perpendicular if $a_1a_2 + b_1b_2 = 0$
3. Two lines will be equal/coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example: Find the point of intersection of the lines 2x-3y+1=0 and x+4y-3=0.

Solution:

What are the values of a_1, b_1, c_1, a_2, b_2, and c_2?

Using the values in the above formula:

$x = \frac{5}{11}, y = \frac{7}{11}$.

Example: Find the point of intersection of the lines x+2y-2=0 and 3x-4y+1=0.

Answer: x=6/7 and y=1.

Concurrent lines

If three lines pass through a common point, then these are called as concurrent (figure 15).

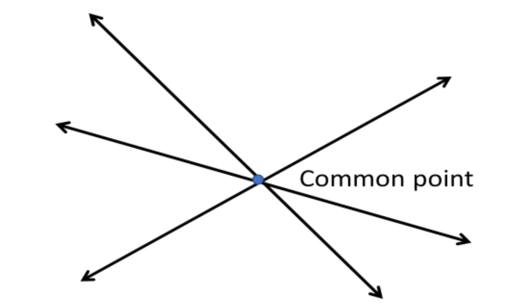


Figure 15

Analytically, the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, and $a_3x+b_3y+c_3=0$ will be coplanar if,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example- Verify whether the lines x-2y+1=0, 2x-4y+2=0 and x+3y+4=0 are concurrent or not!

Method of solution:

Here find the values of a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, and c_3.

Use these values in the above formula. Is the value equal to 0?

System of lines

When we think of more than one line, then it is referred as system of lines. We can draw more than one line in a plane under different conditions viz. 1) Parallel to a given line 2) Perpendicular to a given line 3) Passing through the point of intersection of two lines.

Lines parallel to a given line

As we know that the parallel lines have same slopes, so a system of lines parallel to the line $ax + by + c = 0$ can be drawn by drawing lines keeping their slopes same. Hence the equation of the line will be of the form $ax + by + k = 0$ where k is a parameter.

Lines Perpendicular to a given line

As we know that for the perpendicular lines the product of the slopes is – 1, so a system of lines perpendicular to the line $ax + by + c = 0$ will be parallel to each other. Hence the slope of the line can be taken as $\frac{b}{a}$ .Hence the equation of the line will be of the form b$x - ay + k = 0$ where k is a parameter.

Lines passing through the point of intersection of two lines

The equation of a line passing through the point of intersection of two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is:

$$a_1x+b_1y+c_1+k\,(a_2x+b_2y+c_2) = 0$$

where k is a parameter.

Perpendicular distance of a point from a line

Let us consider a point P (x_1, y_1) in X-Y plane.

We have got a line L_1: ax +by +c=0 already in the plane. When we are asked to find the perpendicular distance from the point to the line, what we understand?

Please consider the figure 16.

We can draw lines from point A to the line L.
How many can you say?
If you think carefully and go on drawing lines from A to line L, it will be uncountable, yes, infinity.
You can draw the lines on this figure itself.
Also you can notice, each line drawn from point A to the given line L makes a different angle with L.
How many lines make exactly 90^o. It is only one.
See the figure 17.
We shall be using an established formula to find the length of perpendicular AB.

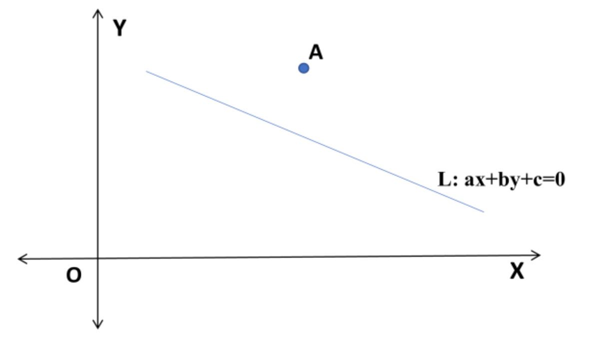


Figure 16

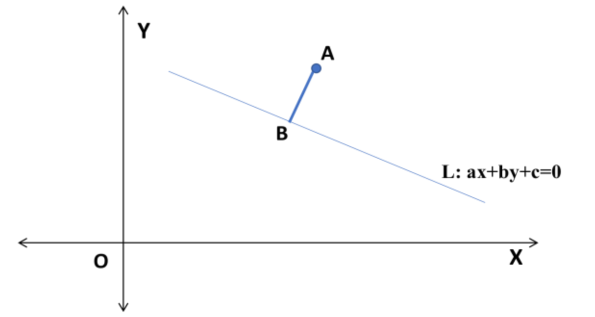


Figure 17

Formula:

Formula:

The length of the perpendicular drawn from the point (x_1, y_1) on the line $ax + by + c = 0$ is:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Proof:

Referring the figure 18, Let L: $ax + by + c = 0$

A be the point (x_1, y_1). Draw a line through the point (x_1, y_1), which is

parallel to L. Let OA=distance of the line L_1 from origin=p_1, OB= distance of the line L from origin=p. The equation of the line L_1 will be $ax + by + k = 0$. Since this line passes through the point (x_1, y_1) therefore, $ax_1 + by_1 + k = 0$

Now $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$ and $p_1 = \frac{-k}{\pm\sqrt{a^2+b^2}}$.

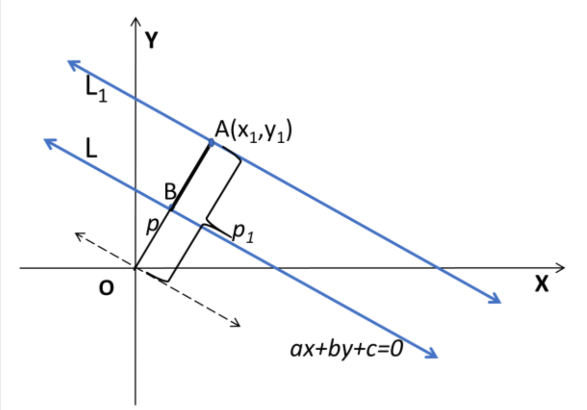


Figure 18

The distance of the point (x_1, y_1) from the line $ax + by + c = 0$ is:

$$AB = |p_1 - p|$$

$$=\left|\frac{-k}{\pm\sqrt{a^2+b^2}} - \frac{-c}{\pm\sqrt{a^2+b^2}}\right|$$

$$=\left|\frac{-k-(-c)}{\pm\sqrt{a^2+b^2}}\right| = \frac{|-k+c|}{\sqrt{a^2+b^2}}$$

$$= \frac{|c-(-ax_1-by_1)|}{\sqrt{a^2+b^2}}$$

$$= \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Example: Find the distance of the point (-1, 2) from the line 5x-2y+1=0.

Method of solution:

What are x_1, and y_1?

What are a, b, and c?

Use these values in the formula!

Is it $\frac{8}{\sqrt{29}}$?

Example: Find the distance of the point (-3, 0) from the line 2x-3y+5=0.

Solution:

Here x_1=-3, y_1=0

a=2, b=-3, c=5

Distance=$\frac{|2\times(-3)+(-3)\times 0+5|}{\sqrt{2^2+(-3)^2}}$=$\frac{1}{\sqrt{13}}$ (answer)

> Note:
> The distance between the parallel lines
> $ax + by + c_1 = 0, and\ ax + by + c_2 = 0$ is:
> $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

Try the following:

1. Find the equation of the line with slope 1/4 and y-intercept of 9.
2. Find the equation of the line passing through the points (-4, 7) and 2,-1).
3. Find the equation of the line passing through the point (-8, 11) and slope 1.
4. Find the equation of the line passing through the point (-2, -1) and inclination of 30^0.
5. Find the equation of the line whose x-intercept and y-intercept are 8 and 3 respectively.
6. Find the equation of the line which cuts x-axis at (2, 0) and y-axis at (0, -7).
7. Find the equation of the line whose sum of the intercepts is 2 and passes through the point (-2,3).
8. Find the equation of the line whose perpendicular distance from origin is 7 and the normal makes an angle 150^0 with positive direction of x-axis.
9. Find the equation of the line passing through (9, -4) and parallel to the line 2x-3y=12.
10. Find the equation of the line passing through (-4, 5) and perpendicular to the line 4x-3y=2.
11. Find the normal form of the equation of the line 4x-3y+11=0.

CIRCLE

In the preceding chapter we have studied various forms of equation of straight lines. In the current chapter we shall discuss a new curve in a plane called as circle.

Loop: Look at the following diagrams. There are 5 pictures. Can you find the area of each of them? We can measure the area of the diagrams 2, 3, 4, 5 only. All the diagrams except 1 is called a closed curve. We can call this as a loop. The diagrams except 1 are called as loops. So loop is a closed region in a plane bounded by curves.

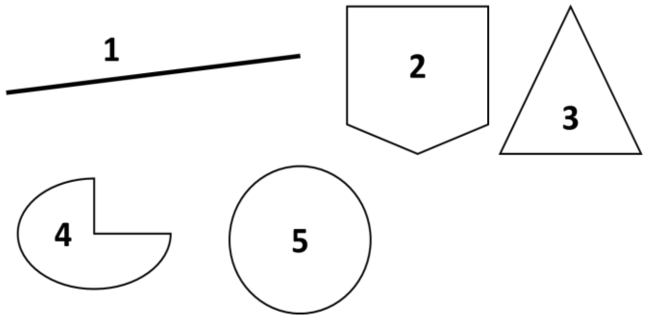


Figure 1

A circle is a type of curve.
Imagine a straight line segment.
As you know a line segment has two ends.
Imagine the line segment that is bent around until both the ends join. We will get a loop. Now let us adjust the loop in such a way that any point on that line will maintain a fixed distance from a point lying inside the loop. In the figure 2, C is a point inside the loop. You can see the relation between the distances of C from any point on the loop. Here CA_1, CA_2, CA_4 are of different lengths.

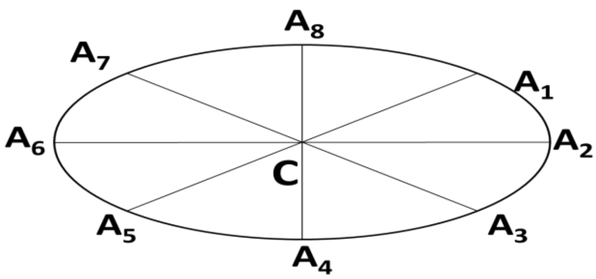
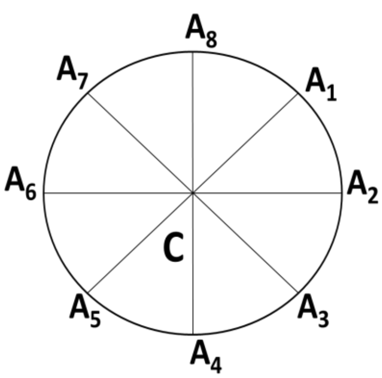


Figure 2



Look at the figure 3. Again you can see see the relation between the distances of C from any point on the loop. Here lines CA_1 to CA_8 are of same lengths. The loop in figure 3 is called as a circle.

Definition of a circle:

Circle is the locus of a point on the loop such that the distance of the point from a fixed point inside the loop is always same. The point fixed point C is called as the center of the circle and the fixed distance is called as radius of the circle.

Properties of a circle

A circle has a center, radius, diameter, tangent, chord, circumference, area.

Figure 3

Equation of circle in different forms:

The equation of a circle can be found subject to the following conditions:

1) Center of the circle and radius is given.
2) Center and a point on the circle are given.
3) Any three points on the circle are given.
4) The end points of a diameter are given.

Case 1 Standard form (center-radius formula)

If you are given center of a circle at (-6, 5) and radius as 3, then can you find the equation of the circle?

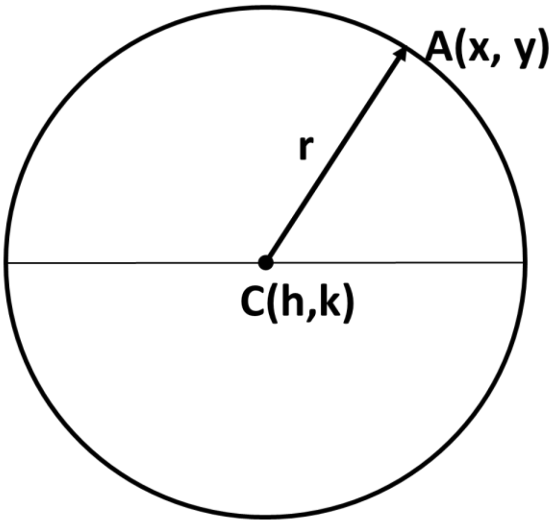
We shall be using a formula.

Formula: The equation of the circle with center at (h, k) and radius 'r' is given by:

$$(x-h)^2+(y-k)^2=r^2$$

Proof:

Referring figure 4, to get the equation of the circle, let us take a variable point A(x, y) on the circle. Using the distance formula between the points C(h, k) and A(x, y) we have the following:



$|CA|=r$

$\Rightarrow \quad \sqrt{(x-h)^2+(y-k)^2}=r$

$\Rightarrow \quad (x-h)^2+(y-k)^2=r^2$

(Proved)

Using this formula let us find the answer to the question asked above.

What are h, k and r?

(Answer; $x^2+y^2+12x-10y+52=0$)

Case 2 Equation of a circle on a diameter

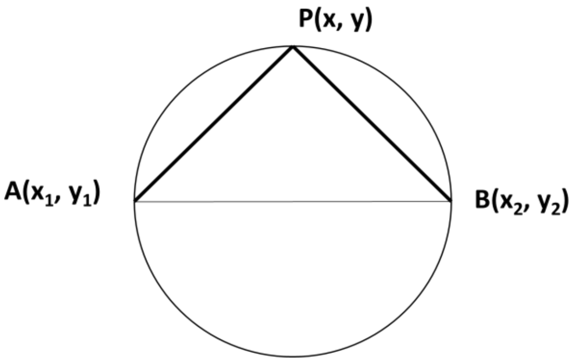
Now if you are given two end points of a diameter of a circle i.e. (-3, 2) and (4, -5) then can you find the equation?

There is a simple formula called as circle on a diameter.

Formula:

The equation of a circle whose end points of a diameter at (x_1, y_1) and (x_2, y_2) is:

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$



As discussed in the straight line chapter, the two lines are perpendiculars if product of their slopes is -1. Here, the end points of a diameter of the circle are given at A (x_1, y_1) and B (x_2, y_2). To find the equation let us take a variable point P(x, y) on the circle.

Now AP is perpendicular to BP (as the angle inscribed in a semi-circle is a right angle).

Hence, Slope of AP × Slope of BP = -1.

$\Rightarrow \quad \frac{y-y_1}{x-x_1}\times\frac{y-y_2}{x-x_2}=-1$

$\Rightarrow \quad (y-y_1)(y-y_2)=-1(x-x_1)(x-x_2)$

$\Rightarrow \quad (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ (Proved)

Now one can solve the question asked above.

Solution:

What are x_1, y_1 and x_2, y_2?

Use formula discussed in case 2.

The question expects answer as: $x^2 + y^2 - x + 3y - 22 = 0$.

Case 3 General equation of circle and its center, radius

If you simplify the equation of circle given in case 1 and case 2 you get a common form as:

$x^2+y^2+(----)x+(----)y+(----)=0$. The terms inside the brackets can be taken to memorize the equation as:

$x^2+y^2+2gx+2fy+c=0$

Formula:

An equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ which is of 2nd degree in x and y is called as a general equation of a circle in which the term "xy" is absent.

Proof:

Let us take the equation of the circle in standard form as

$(x - h)^2 + (y - k)^2 = r^2$, Where the center is at C(h, k) and radius as r.

Expanding the terms, we get

$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

Alternatively, we can write this in the simplest form as:

$x^2 + y^2 + 2gx + 2fy + c = 0$

Where, -h=g, -k=f and $c = h^2 + k^2 - r^2$

In other words, h=-g, k=-f and $r^2 = h^2 + k^2 - c = (-g)^2 + (-f)^2 - c = g^2 + f^2 - c$

Hence, the coordinates of the center will be at (-g, -f) and radius = r=$\sqrt{g^2 + f^2 - c}$.

Notes:

1. The equation of a circle with centre at origin and radius r will be: $x^2 + y^2 = r^2$.
2. The equation of a circle with centre at (h, k) and touching x-axis shall be:

   $(x - h)^2 + (y - k)^2 = k^2$
3. The equation of a circle with centre at (h, k) and touching y-axis shall be:

   $(x - h)^2 + (y - k)^2 = h^2$
4. The equation of a circle with radius r and touching both axes will be:

   $(x - r)^2 + (y - r)^2 = r^2$
5. In the general equation the coefficients of x^2 and y^2 are either same or unity.
6. If the center is on X-axis , then the equation of the circle will be

   $$x^2 + y^2 + 2gx + c = 0$$
7. If the center is on Y-axis , then the equation of the circle will be

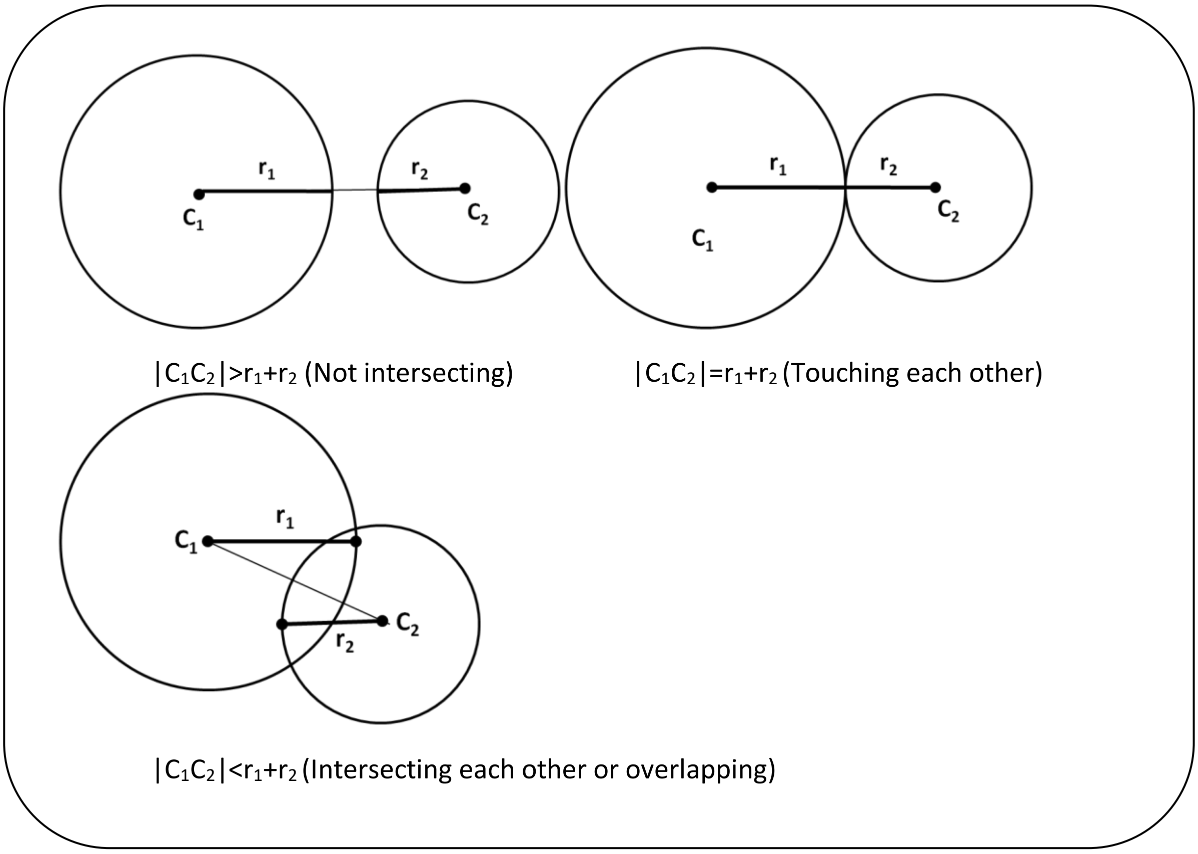
   $$x^2 + y^2 + 2fy + c = 0$$

Example: Find the center and radius of the circle $x^2+y^2-8x+10y-12=0$.

Solution:
What are the values of g, f and c?
(Answer (4, -5) and $r=\sqrt{43}$)

Example: Find the equation of the circle with centre (–3, 2) and radius 4.

Solution:
Here $h = -3$, $k = 2$ and $r = 4$.
Therefore, the equation of the circle can be obtained by using the center-radius formula.
Hence, the equation will be:
$(x + 3)^2 + (y - 2)^2 = 16$ (simplify).
See the diagram below



Circle passing through three points
Can you get the circle passing through the points (x_1, y_1) , (x_2, y_2) and (x_3, y_3).
Yes, the equation can be obtained by using the general equation of the circle because the equation contains three unknown constants. To get the values of the unknown constants viz., g, f, c

Put the values x=x1, y=y1, x=x2, y=y2, and x=x3, y=y3 separately in the general equation you will get three linear equations containing g, f, and c. Now solve these three equations by any method to get g, f, and c. Finally use all these values in the general equation to get the equation of the circle.

Example: Find the equation of the circle passing through the points (-1, 0), (2, 1), and (-1, 3).

Solution:
Let the equation of the circle in general form be:
$x^2 + y^2 + 2gx + 2fy + c = 0$ -------- (1)
Since the circle passes through the points (-1, 0), (2, 1), and (-1, 3), so these points will satisfy the equation of the circle.
Hence, for the point (-1, 0),
The equation reduces to 1-2g+c=0 ------- (2)
For the point (2, 1)
4g+2f+c+5=0 ------- (3)
And for the point (-1, 3)
-2g+6f+c+10=0 ------- (4)
Eqn(4) – Eqn(3)$\Rightarrow$ -6g+4f+5=0 ------- (5)
Eqn(3) – Eqn(2)$\Rightarrow$ 6g+2f+4=0 ------- (6)
Eqn(5) + Eqn(6)$\Rightarrow$ 6f+9=0 $\Rightarrow$ f=-9/6=-3/2
Using this value in Eqn(6) g=-(2f+4)/6=-1/6
Using this value in Eqn(2) c=2g-1=-4/3
Putting all these values in Equation (1) we get the circle as:
$x^2 + y^2 + 2\left(\frac{-1}{6}\right)x + 2\left(\frac{-3}{2}\right)y - \frac{4}{3} = 0$
$\Rightarrow 6x^2 + 6y^2 - 2x - 18y - 8 = 0$ (Answer)

Example: Find equation of the circle whose center lies on X-axis and passing through the points (3, 2) and (1, -4).

Solution:
Let the equation of the circle in general form be:
$x^2 + y^2 + 2gx + 2fy + c = 0$
Since the center of the circle lies on X-axis, therefore f=0.
Hence, the equation reduces to
$x^2 + y^2 + 2fy + c = 0$ -------- (1)
Since the circle passes through the points (3, 2) and (1, -4) therefore,
The equation (1) reduces to
4f+c+13=0 -------- (2) for (3, 2)
and -8f+c+17=0 -------- (3) for (1, -4)
Eqn(3) – Eqn(2) $\Rightarrow$ f=1/3
Using this value in Eqn(3), c=8f-17=43/3.
Putting all these in equation (1) the circle will be:
$x^2 + y^2 + \frac{2}{3}y + \frac{43}{3} = 0$
$\Rightarrow 3x^2 + 3y^2 + 2y + 43 = 0$ (Answer)

Question:

1 Find the equation of the circle with centre at (1, 2) and radius 3.

2 Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

3 Find the equation of the circle passing through the points (4, 1) and (6, 0) and whose centre is on the line $4x + y = 16$.

4 Find the equation of a circle whose ends of a diameter are at (2, 4) and (-3, 1).

5 Find the center and radius of the circle $x^2 + y^2 - 4x - 8y - 61 = 0$.

CHAPTER-4
Co-ordinate Geometry (3D)

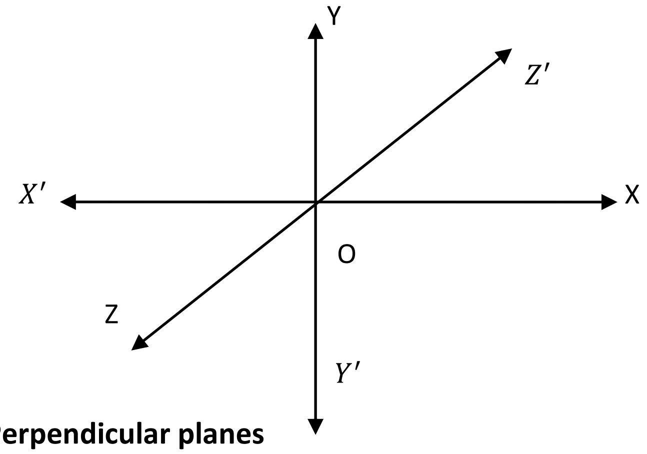
In two dimensional co-ordinate geometry, the position of a point in a plane is determined with respect to two intersecting lines. These lines are called axes and the point of intersection of the lines is called origin. It is also called rectangular axes since the angle between two lies is 90^0.The axes devide the plane into four quadrants.

In Three dimensional geometry we have three mutually perpendicular lines and divide the space in eight equal parts, each part is called octant. Also gives three mutually perpendicular planes. The lines are called co-ordinate axes.

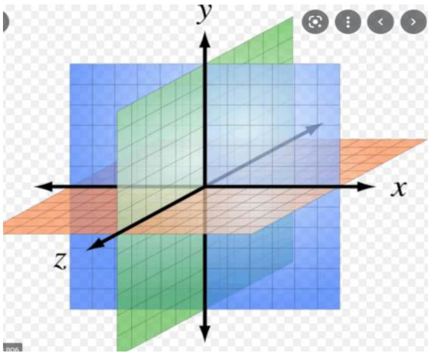
Co-ordinate axes and co-ordinate planes:

Let XOX', YOY' and ZOZ' are three mutually perpendicular lines intersecting at O. Here O is called origin and the lines XOX', YOY' and ZOZ' are called x-axis, y-axis and z-axis respectively. These three lines are called the rectangular axes of co-ordinates. The three mutually perpendicular planes are XY, YZ and ZX plane.OX,OY and OZ are taken as positive direction where as OX', OY' and OZ' are the negative direction of x-axis, y-axis and z-axis.

Rectangular Coordinate System



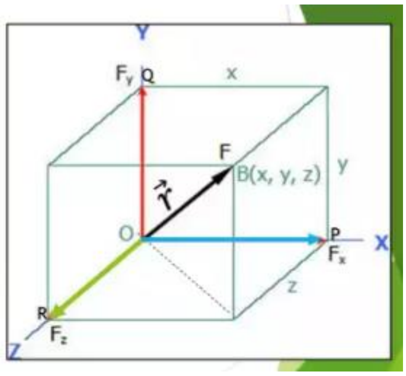
Three Mutually Perpendicular planes



Brown:XZ,Green:YZ,Blue:XY

Co-ordinate of a point on a space:

Let P be any point on the space. Draw perpendiculars from P on XY, YZ and ZX plane, which meets at L, M and N respectively. So the length of PM=x,PN=y and PL=z represents the co-ordinate of the point P, which is written as P(x,y,z),(which is read as P be a point having co-ordinate (x,y,z).



Note:

1.If a point P(x,y,z) lies on XY plane then z=0 ,so the co-ordinate of the point is P(x,y,0) and the equation of XY plane is z=0.

If a point P(x,y,z) lies on YZ plane then x=0 ,so the co-ordinate of the point is P(0,y,z) and the equation of YZ plane is x=0.

If a point P(x,y,z) lies on ZX plane then y=0 ,so the co-ordinate of the point is P(x,0,z) and the equation of ZX plane is y=0.

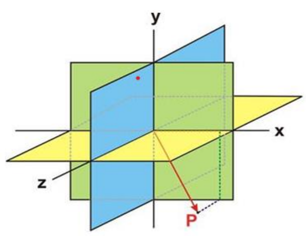
2. If a point P(x,y,z) lies on X-axis then y=z=0 ,so the co-ordinate of the point is P(x,0,0) and the equation of x-axis is y=0 and z=0.

If a point P(x,y,z) lies on y-axis then x=z=0 ,so the co-ordinate of the point is P(0,y,0) and the equation of y-axis is x=0 and z=0.

If a point P(x,y,z) lies on z-axis then x= y=0 ,so the co-ordinate of the point is P(0,0,z) and the equation of z-axis is x=0 and y=0.

Sign of different octants:

Co-ordinate/Octants	OXYZ	$OXYZ'$	$OXY'Z$	$OX'YZ$	$OXY'Z'$	$OX'YZ'$	$OX'Y'Z$	$OX'Y'Z'$
x y z	+ + +	+ + -	+ - +	- + +	+ - -	- + -	- - +	- - -



Distance Formula:

Theorem: The distance between any two points $P(x_1, y_1, z_1) amd\ Q(x_2, y_2, z_2)$ is given by

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Proof: Let $P(x_1, y_1, z_1) amd\ Q(x_2, y_2, z_2)$ $are\ two\ points\ .Draw\ perpendiculars\ from\ P\ and\ Q$ $on\ ZX\ plane\ , which\ meets\ at\ L\ and\ M\ respectively\ , so\ the\ co-ordinates\ of\ L\ and\ M$ $are\ L(x_1, 0, z_1) and\ M(x_2, 0, z_2)$.Again draw a perpendicular from P on QM which meets at R

Here $PL = y_1\ and\ QM = y_2, so\ RQ = y_2 - y_1$ Figure:

Now $LM = PR = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}$

In the triangle PRQ

$PQ^2 = PR^2 + RQ^2 = (x_2 - x_1)^2 + (z_2 - z_1)^2 + (y_2 - y_1)^2$

$\Rightarrow PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Note: The distance from origin to any point $P(x, y, z) is\ OP = \sqrt{x^2 + y^2 + z^2}$

Note:

1.The distance from any point $P(x, y, z)\ on\ x - axis\ , y - axis\ and\ z - axis\ is\ given\ by$

$$PL = \sqrt{y^2 + z^2}$$

$$PM = \sqrt{x^2 + z^2}$$

$$PN = \sqrt{x^2 + y^2}$$

2. The distance from any point $P(x, y, z)\ on\ xy\ , yz\ and\ zx\ plane\ is\ given\ by$

$$PL = \sqrt{z^2} = |z|$$

$$PM = \sqrt{x^2} = |x|$$

$$PN = \sqrt{y^2} = |y|$$

Division Rule:

(i) Internal Division :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and R be any point on the line segment joining P and q such that it divides PQ internally in the ratio $m:n$

Then coordinates of R are: $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

(ii) External Division :

In above case if R divides externally in the ratio $m:n$, then coordinates of R are

$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$

(iii) Mid point Rule :

If R is the mid point of PQ, then coordinates of R are

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Image of a point with respect to plane:

Image of a point $P(x, y, z)$ w.r.t XY plane is $(x,, y, -z)$

Image of a point $P(x, y, z)$ w.r.t YZ plane is $(-x,, y, z)$

Image of a point $P(x, y, z)$ w.r.t ZX plane is $(x,, -y, z)$

Direction Cosines and Direction Ratios of a line:

Let the line OP makes angles α, β, γ with OX,OY and OZ respectively.Then cos α ,

cos β and cos γ are called the *direction cosines*(d.cs) of the line OP

The d.cs of a line are denoted by l, m, n

$i.e.\ l = \cos\alpha, m = \cos\beta, n = \cos\gamma$

Direction cosines of a line always satisfy the relation : $l^2 + m^2 + n^2 = 1$

Choose three real numbers a, b, c such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$,then these numbers a, b, c

are called *direction ratios* of the line

To find direction cosines of a line when direction ratios given:

Let a, b, c are direction ratios of the given line.Then its direction cosines are given by

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

Direction ratios of a line joining two points:

The direction ratios of the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$\langle\ x_2 - x_1, y_2 - y_1, z_2 - z_1\ \rangle$$

Angle between two lines:

(i) The angle between two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\theta = \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

(ii) The angle between two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 is

$$\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$$

Condition for Parallel:

In the above case, if two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (in case of d.rs given)

and $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ (in case of d.cs given)

Condition for Perpendicular:

In the above case, if two lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (in case of d.rs) and $l_1l_2 + m_1m_2 + n_1n_2 = 0$ (in case of d.cs given)

Ex-1: Find the distance of the point P(3,4,5) from x-axis

Sol^n: The coordinates of the foot of the perpendicular from P(3,4,5) on x-axis are Q(3,0,0)

Therefore the required distance = $\sqrt{(3-3)^2 + (4-0)^2 + (5-0)^2} = \sqrt{41}$

Ex-2: Find the distance of the point $P(x, y, z)$ from z-axis

Sol^n**:** The distance of the point $P(x, y, z)$ from z-axis ,i.e. from the point (0,0,z) is $\sqrt{x^2 + y^2}$

Ex-3: Determine the distance of the point (a,b,c) from zx-plane

Sol^n: The distance of the point (a,b,c) from zx-plane, i.e. from the point $(a,0,c)$ is

$\sqrt{(a-a)^2+(b-0)^2+(c-c)^2}=b$

Ex – 4: If l_1, m_1, n_1 and l_2, m_2, n_2 are dcs of two $\perp_r$ lines the, find the value of $l_1l_2+m_1m_2+n_1n_2$

Sol^n : $l_1l_2+m_1m_2+n_1n_2=0$

Ex - 5: If the distance between points (-1,-1,z) and (1,-1,1) is 2, find the value of z

Sol^n : From distance formula,

$$(1+1)^2+(-1+1)^2+(1-z)^2=2^2$$
$$\Rightarrow 4+0+(1-z)^2=4$$
$$\Rightarrow (1-z)^2=0, \ \Rightarrow z=1$$

Ex - 6: Find the image of the point (6,3,-4) with respect to yz - plane

Sol^n : (-6,3,-4)

Ex - 7: If $\frac{2}{7}, \frac{3}{7}, \frac{k}{7}$ represents dcs of a line, find the value of k

Sol^n : For $\frac{2}{7}, \frac{3}{7}, \frac{k}{7}$ dcs of a line, $\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{k}{7}\right)^2=1, \Rightarrow k=\pm 6$

Ex - 8: Find the ratio in which the line joining points (2,4,5) and (3,5,-4) is divided by xy - plane

Sol^n : Let the ratio be $\lambda:1$, given points P(2,4,5) and Q(3,5,-4)

By division formula, coordinates of division point are $\left(\frac{3\lambda+2}{\lambda+1}, \frac{5\lambda+4}{\lambda+1}, \frac{-4\lambda+5}{\lambda+1}\right)$

On xy - plane, $z=0 \Rightarrow \frac{-4\lambda+5}{\lambda+1}=0, \Rightarrow \lambda=\frac{5}{4}$

Hence required ratio is 5 : 4

Ex - 9: Find foot of the perpendicular drawn from the point (-1,3,4) on yz - plane

Sol^n : On yz - plane $x=0$. Thus foot of the $\perp_r$ is (0,3,4)

Ex -10 Find coordinates of foot of the perpendicular drawn from the point A(1,1,1) on the line joining points B(1,4,6) and C(5,4,4)

Sol^n : Let P be the foot off the perpendicular from A on BC, divide BC in the ratio $k:1$

$\therefore$ Coordinates of P are $\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$ ....(i)

Direction ratios of BC are (5-1,4-4,4-6) i.e. (4,0,-2)

Direction ratios of AP are

$$\left(\frac{5k+1}{k+1}-1, \frac{4k+4}{k+1}-1, \frac{4k+6}{k+1}-1\right)$$

i.e. $\left(\frac{4k}{k+1}, \frac{3k+3}{k+1}, \frac{3k+5}{k+1}\right)$

Since AP $\perp_r$ BC, $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow 4.\frac{4k}{k+1}+0.\frac{3k+3}{k+1}+(-2).\frac{3k+5}{k+1}=0$

$\Rightarrow 16k-6k-10=0, \Rightarrow k=1$

Hence coordinates of the point P(3,4,5) (*from* (i))

Ex -11: Show that the points A(3,2,4), B(4,5,2) and C(5,8,0) are collinear

Sol^n : Direction ratios of line AB are $\langle 4-3, 5-2, 2-4 \rangle$ *i.e.* $\langle 1,3,-2 \rangle$

Direction ratios of line BC are $\langle 5-4, 8-5, 0-2 \rangle$ *i.e.* $\langle 1,3,-2 \rangle$

$\because \frac{1}{1}=\frac{3}{3}=\frac{-2}{-2}$ $\Rightarrow$ AB and BC are parallel

But point B is common to both

Hence points A, B, C are collinear

Ex -12 : Find angle between two diagonals of a cube

Sol^n : Consider the cube OABCDEFG with each side length a

Now the coordinates of the vertices of the cube are O(0,0,0), A$(a,0,0)$, B$(a,0,a)$, C$(0,0,a)$, D$(0,a,a)$, E$(0,a,0)$, F$(a,a,0)$, G(a,a,a)

Take two diagonals OG and AD

Direction ratios of OG are $\langle a-0, a-0, a-0 \rangle$ *i.e,* $\langle a,a,a \rangle$

Direction ratios of AD are $\langle 0-a, a-0, a-0 \rangle$ *i.e.* $\langle -a,a,a \rangle$

$$\therefore \text{Angle between diagonal} = \theta = \cos^{-1}\left(\frac{a.-a+a.a+a.a}{\sqrt{a^2+a^2+a^2}.\sqrt{a^2+a^2+a^2}}\right)$$

$$=\cos^{-1}\left(\frac{a^2}{3a^2}\right)=\cos^{-1}\left(\frac{1}{3}\right)$$

.Assignment-1:

1.Find the distance between the points $(-2,4,1)\ and\ (1,2,-5)$.

2. Show that the points $(0,1,2),(2,-1,3) and\ (1,-3,1) are\ the\ vertices\ of\ an\ isosceles$ right angled triangle.

3.Prove that the triangle with vertices $(1,2,3),(2,3,1) and\ (3,1,2) are\ formed\ an\ equilateral\ triangle$

4.Show that the points $(3,3,3),(0,6,3),(1,7,7) and\ (4,4,7) are\ the\ vertices\ of\ a\ square$.

5.Find the value of z if the distance between the points $(2,-3,1) and\ (2,1,z) is\ 5\ units$.

6. Find the distance from the point $(2,-3,1) on\ x-axis$.

7.Find the distance from the point $(\alpha,\beta,\gamma) on\ YZ\ plane$.

8.Prove by using distance formula that the points $P(1,2,3),Q(-1,-1,-1) and\ R(3,5,7) are\ collinear$.

9. Find the image of the point $(6,3,-4) with\ respect\ to\ YZ\ plane$.

10.Find the perimeter of a triangle whose vertices are $(0,1,2),(2,0,4) and\ (-4,-2,7)$.

11.Find the co-ordinates of a point which divides the line segment joining $(1,3,7) and\ (6,3,2) in\ the$ ratio 2:3.

12.Find the ratio in which the line segment joining the points $(4,4,-10) and\ (-2,2,4) is\ divided$ by YZ plane.

13. Find the ratio in which the line segment joining the points $(2,4,5) and\ (3,5,-4)\ is\ divided$ by XY plane.

14. Find the ratio in which the line segment joining the points $(4,4,-10) and\ (-2,2,4) is\ divided$ by plane $x+y+z=3$.

15. Find the mid point on the line joining the points $(2,-1,3) and\ (3,1,5)$.

Assignment-2:

1.Show that the points $(3,2,4),(4,5,2)\ and\ (5,8,0) are\ collinear$.

2.Find the value of k such that the points $(1,-2,3),(3,-1,2) and\ (7,1,k)\ are\ colinear$.

3.If a line perpendicular to z-axis and makes angle 60^0 with x-axis. Find the angle it makes with y-axis.

4.Find the projection of the line segment joining $(1,3,-1) and\ (3,2,4)$ on z-axis.

5.Find the image of the point $(6,3,-4)$ with respect YZ plane.

6.Find the dcs of a line passing through the points $(0,0,0)\ and\ (1,2,3)$.

7.Find the co-ordinates of the point where the perpendicular from the point $(1,1,1)$ on the line joining the points $(-9,4,5)\ and\ (1,0,-1)$.

8.Find the angle between two main diagonals of a cube.

9. Find the angle between a main diagonal and one edge of a cube.

10. Find the acute angle between the lines whose drs are $< 1,1,2 >$ and $< \sqrt{3} - 1, -\sqrt{3} - 1,4 >$.

11. Find the angle between the lines joining the points $(1,4,2), (-2,1,2) and\ (1,2,3), (2,3,1)$.

12. Show that the points $(0,0,0), (3,4,5)\ and\ (-3,-4,-5) are\ collinear$.

13.The projection of a line segment on x-axis, y-axis and z-axis are 4,12,3 . Find the length of the line segment and its dcs.

14. Find the co-ordinates of the foot of the perpendicular from the point $(1,2,3)$ on the line joining the points $(-2,3,4)\ and\ (2,-1,6)$.

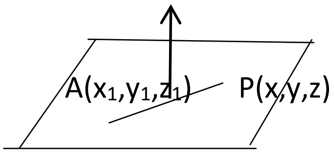
15. Find the image of the point $(-2,3,1)$ with respect XY plane.

PLANE

A plane is a surface such that the line joining any two points on the surface lies wholly on it
Every general equation of the first degree in x, y, z of the form $\text{A}x + \text{B}y + \text{C}z + \text{D} = 0$
represents a plane where A, B, C are drs of the normal to plane

Equation of a plane passing through a given point when direction cosines of a normal are given:

L(l,m,n)



Let $\text{A}(x_1, y_1, z_1)$ be a given point on the plane and AL be normal to the plane whose direction cosines be l, m, n.Take any point $\text{P}(x, y, z)$ on the plane.Henc e AP is perpendicu lar to AL
Now direction rarios of AP are $\langle x - x_1, y - y_1, z - z_1 \rangle$
Using perpendicu larity condition, we have

$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$, which is the required equation of the plane

Note : If $\langle a, b, c \rangle$ are direction ratios of the normal to the plane, then equation of plane passing through th e point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Ex -1: Find the direction cosines of the normal to the plane $x - y + 2z - 3 = 0$

Sol^n : Direction ratios of the normal to the plane are $\langle 1,-1,2 \rangle$

The direction cosines are

$$\frac{1}{\sqrt{1^2+(-1)^2+2^2}}, \frac{-1}{\sqrt{1^2+(-1)^2+2^2}}, \frac{2}{\sqrt{1^2+(-1)^2+2^2}}$$

$$i.e. \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

Ex - 2 : Find equation of plane passing through t he point (2,3,1), the direction ratios of the normal to the plane being $\langle 3,5,7 \rangle$

Sol^n : Here $\langle a,b,c \rangle = \langle 3,5,7 \rangle$ and $(x_1, y_1, z_1) = (2,3,1)$

Required equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\Rightarrow 3(x-2) + 5(y-3) + 7(z-1) = 0$$

$$\Rightarrow 3x + 5y + 7z - 28 = 0$$

Note : Equation of xy plane is $z = 0$

Equation of yz plane is $x = 0$

Equation of zx plane is $y = 0$

Equation of plane through non - collinear points :

Let three given non - collinear points are $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3)

Then equation of plane passing through above points is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Ex - 3 : Find equation of plane passing through points $(1,1,0), (-2,2,-1), (1,2,1)$

Sol^n : Required equation of plane is

$$\begin{vmatrix} x-1 & y-1 & z-0 \\ -2-1 & 2-1 & -1-0 \\ 1-1 & 2-1 & 1-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ -3 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0, \quad \Rightarrow (x-1)(1+1)-(y-1)(-3-0)+z(-3-0)=0$$

$$\Rightarrow 2(x-1)+3(y-1)-3z=0$$

$$\Rightarrow 2x+3y-3z-5=0$$

Intercept form :

Let the intercepts on x, y and z - axes be a, b, c respective ly.Hence the plane meets the coordinate axes at $(a,0,0), (0,b,0)$ and $(0,0,c)$ respective ly.

$\therefore$ Equation of plane through t hese points is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0, \Rightarrow xbc + yac + zab = abc$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

which is required equation of plane in intercept form

Ex - 4 : Find equation of plane whose intercepts on axes are $2, -1, 5$ on x, y and z - axis respective ly

Sol^n : Equation of plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \Rightarrow \frac{x}{2} + \frac{y}{-1} + \frac{z}{5} = 1$$

$$\Rightarrow \frac{5x - 10y + 2z}{10} = 1$$

$$\Rightarrow 5x - 10y + 2z - 10 = 0$$

Ex - 5 : Find equation of plane passing through the point (-1,3,0) and perpendicular to the planes $x+2y+2z-5=0$ and $3x+3y+2z-8=0$

Sol^n : Any plane passing through the point (-1,3,0) is given by $a(x+1)+b(y-3)+c(z-0)=0$

Since this plane is perpndicular to planes $x+2y+2z-5=0$ and $3x+3y+2z-8=0$

by condition of perpendicularity, we have

$$a+2b+2c=0$$

and $$3a+3b+2c=0$$

Solving by method of cross multiplication, we have

$$\frac{a}{4-6}=\frac{b}{6-2}=\frac{c}{3-6}, \Rightarrow \frac{a}{-2}=\frac{b}{4}=\frac{c}{-3}$$

$\therefore$ Equation of required plane is

$$-2(x+1)+4(y-3)-3(z-0)=0$$

$$\Rightarrow -2x+4y-3z-14=0, \Rightarrow 2x-4y+3z+14=0$$

Ex - 6 : Find equation of plane passing through the points (2,2,1) and (9,3,6) and perpendicular to the plane $2x+6y+6z+9=0$

Sol^n : Any plane passing through the point (2,2,1) is given by $a(x-2)+b(y-2)+c(z-1)=0$

Since it also passes through the point (9,3,6), we have

$a(9-2)+b(3-2)+c(6-1)=5$, $\Rightarrow 7a+b+5c=0$ ....(i)

Also the above plane is perpendicular to the plane $2x+6y+6z+9=0$

so by perpendicular condition , $2a+6b+6c=0$ ....(ii)

Now solving equation (i) and (ii) by method of cross multiplication, we have

$$\frac{a}{6-30}=\frac{b}{10-42}=\frac{c}{42-2} \Rightarrow \frac{a}{-3}=\frac{b}{-4}=\frac{c}{5}$$

Hence equation of required plane is

$$-3(x-2)-4(y-2)+5(z-1)=0$$

$$\Rightarrow -3x-4y+5z+9=0, \Rightarrow 3x+4y-5z-9=0$$

<u>Normal form of the equation of a plane</u> :

Let p be the length of perpendicular drawn from origin to the plane and l,m,n be the direction cosines of the normal.Then equation of the plane is given by

$$lx+my+nz=p$$

<u>Transformation of the general equation of a plane to the normal form</u> :

Normal form of the general form of plane $ax+by+cz+d=0$ is given by

$$\frac{a}{\sqrt{a^2+b^2+c^2}}.x+\frac{b}{\sqrt{a^2+b^2+c^2}}.y+\frac{c}{\sqrt{a^2+b^2+c^2}}.z=p$$

where $p=\dfrac{-d}{\sqrt{a^2+b^2+c^2}}$

Ex - 7 : Obtain the normal form of the plane $2x-3y+5z+1=0$

Sol^n : Direction ratios of the normal to the given plane are $\langle 2,-3,5 \rangle$

$\therefore$ Direction cosines are $\dfrac{2}{\sqrt{2^2+(-3)^2+5^2}}, \dfrac{-3}{\sqrt{2^2+(-3)^2+5^2}}, \dfrac{5}{\sqrt{2^2+(-3)^2+5^2}}$

$i.e. \dfrac{2}{\sqrt{38}}, \dfrac{-3}{\sqrt{38}}, \dfrac{5}{\sqrt{38}}$

$p=\dfrac{1}{\sqrt{38}}$

$\therefore$ Equation of plane in the normal form is

$$\frac{2x}{-\sqrt{38}}+\frac{3y}{\sqrt{38}}+\frac{5z}{-\sqrt{38}}=\frac{1}{\sqrt{38}}$$

<u>Equation of plane parallel to another plane</u> :

Equation of plane parallel to the given plane $Ax+By+Cz+D=0$ is $Ax+By+Cz+K=0$

where k is constant t o find out

Ex - 8 : Find equation of plane passing through t he point $(1,-2,4)$ and parallel to the plane $x-2y+4z-2=0$

Sol^n : Let equation of parallel plane be $x-2y+4z+k=0$

for it passes through $(1,-2,4), \Rightarrow 1-2.(-2)+4.4+k=0$

$\Rightarrow k=-21$

Hence required equation of plane is $x-2y+4z-21=0$

Angle between two planes :

The angle between two planes is equal to the angle between their normals

Let two planes be

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $$a_2x + b_2y + c_2z + d_2 = 0$$

The direction ratios of the normal to planes are $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$

Hence $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

If above two planes are

(i) perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) identical, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

Ex - 9 : Find the angle between the planes $x+3y-5z+1=0$ and $2x+y+z+2=0$

Sol^n : Here $a_1=1, b_1=3, c_1-5, a_2=2, b_2=1, c_2=1$

$$\theta=\cos^{-1}\left(\frac{1.2+3.1+(-5).1}{\sqrt{1^2+3^2+(-5)^2}.\sqrt{2^2+1^2+1^2}}\right)=\cos^{-1}(0)=\frac{\pi}{2}$$

Ex - 10 : Find the value of k if the planes $x+3y+kz=0$ and $kx+y+2z=0$ are perpendicular to each other

Sol^n : For perpendicular , $a_1a_2+b_1b_2+c_1c_2=0$

$$\Rightarrow 1.k+3.1+k.2=0$$

$$\Rightarrow k=-1$$

Distance of a point from a plane :

(i) The length(distance) of the perpendicular from any point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is given by

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

(ii) The length(distance) of the perpendicular from origin to the plane $ax+by+cz+d=0$ is given by

$$\left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$$

Distance($\perp_r$) between two parallel planes :

Let two parallel planes are

$$ax+by+cz+d_1=0$$

and $$ax+by+cz+d_2=0$$

Then distance between them is given by

$$\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

Ex - 11 : Find the perpendicular distance of the point $(1,-1,-1)$ from the plane $2x+y+2z+4=0$

Sol^n : Required distance

$$=\left|\frac{2.1+1.(-1)+2.(-1)+4}{\sqrt{2^2+1^2+2^2}}\right|=\frac{3}{3}=1$$

Ex - 12 : Find the distance the planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$

Sol^n : $(2x-2y+z+3=0)\times 2 \Rightarrow 4x-4y+2z+6=0$

$$4x-4y+2z+5=0$$

$$\therefore \text{Distance }=\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|=\frac{6-5}{\sqrt{16+16+4}}=\frac{1}{6}$$

Ex -13: Find equation of plane bisecting the line segment joining points $(-1,4,3)$ and $(5,-2,-1)$ at right angles

Sol^n : Let the plane bisects the line segment joining points $A(-1,4,3)$ and $B(5,-2,-1)$ at P

Now P is the midpoint of AB and $P(2,1,1) \rightarrow (x_1, y_1, z_1)$

As per question AB is normal to the plane

Direction ratios of the normal AB are $\langle 5+1, -2-4, -1-3 \rangle$, *i.e.* $\langle 6,-6,-4 \rangle \rightarrow \langle a,b,c \rangle$

Hence rquired equation of the plane is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$\Rightarrow 6(x-2)-6(y-1)-4(z-1)=0$

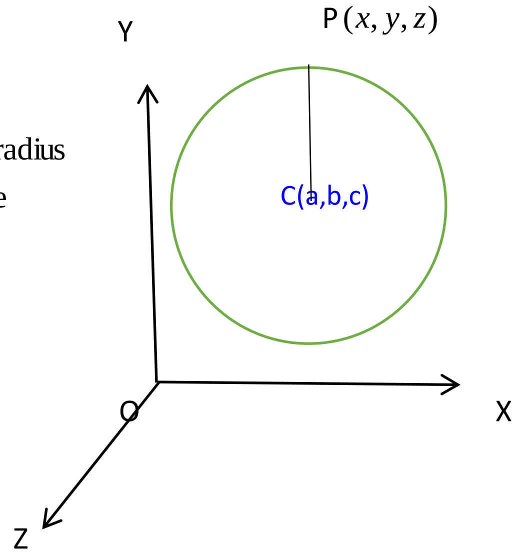
$\Rightarrow 6x-6y-4z-2=0, \Rightarrow 3x-3y-2z-1=0$

Assignment

1. Find equation of plane perpendicular to z-axis and passing through the point $(1,-2,4)$
2. Find equation of plane passing through $(1,1,2)$ and parallel to the plane $x+y+z-1=0$
3. Aplane whose normal has drs $<3,-2,k>$ is parallel to the line joining $(-1,1,-4)$ and $(5,6,-2)$, the find the value of k
4. Find the value of k such that the perpendicular distance of the point $(1,1,1)$ from the plane $2x+y-2z=k=0$ is 1
5. Find angle between the planes $2x-y+z=6$ and $x+y+2z=3$
6. Find the distance between the parallel planes $2x-3y+6z+1=0$ and $4x-6y+12z+5=0$
7. Find equation of plane passing through points $(0,-1,-1),(4,5,1)$ and $(3,9,4)$
8. Find equation of plane passing through the point $(1,-1,2)$ and perpendicular to each of the planes $3x+2y-3z-1=0$ and $5x-4y+z-5=0$
9. Find equation of plane passing through the point $(-1,3,2)$ and perpendicular to each of the planes $x+2y+2z=5$ and $3x+3y+2z=8$
10. Find equation of plane which passes through $(3,4,-1)$ and perpendicular to the line whose direction ratios are $<5,2,-3>$
11. Find equation of plane passing through the points $(1,2,-3)$ and perpendicular to planes $3x-y+2z+3=0$ and $x=3y-2z-1=0$

SPHERE

A sphere is locus of a point in space which is always at a fixed distance from a fixed point The fixed point is called centre of the sphere and the fixed distance is called radius of the sphere



Standard Equation of Sphere :

Let $C(a,b,c)$ be the given centre and r be the given radius of the sphere. Take any point $P(x,y,z)$ on the sphere

Now, $CP = r$

$\Rightarrow \quad CP^2 = r^2$

$\Rightarrow \quad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

which is required equation of sphere

Ex -1: Find the equation of the sphere whose centre at $(2,-3,4)$ and radius is 5

Sol^n : Centre $(2,-3,4)$ and radius is 5

$i.e.\ a=2, b=-3, c=4,\ r=5$

Equation of sphere is

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$\Rightarrow \quad (x-2)^2+(y+3)^2+(z-4)^2=5^2$

$\Rightarrow \quad x^2+y^2+z^2-4x+6y-8z+4=0$

Ex - 2: Find equation of sphere whose centre is $(1,-2,3)$ and which passes through t he point $(0,2,-1)$

Sol^n : Given centre $C(1,-2,3), \Rightarrow a=1, b=-2, c=3$

Radius is the line segment joining points $(1,-2,3)$ and $(0,2,-1)$

$\Rightarrow \quad r=\sqrt{(0-1)^2+(2+2)^2+(-1-3)^2}=\sqrt{33}$

Equation of sphere is

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$\Rightarrow \quad (x-1)^2+(y+2)^2+(z-3)^2=33$

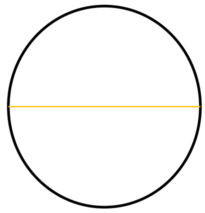
$\Rightarrow \quad x^2+y^2+z^2-2x+4y-6z-19=0$

Ex - 3: Find equation of sphere whose centre at origin and radius is $\sqrt{3}$

Sol^n : Equation of required sphere is

$x^2+y^2+z^2=r^2$

$\Rightarrow \quad x^2+y^2+z^2=3$



Equation of sphere through end points of a diameter :

Let PQ be one diameter of sphere where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let $A(x, y, z)$ be any point on it.

Then required equation of sphere is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

Ex - 4 : Find equation of sphere on join of (2,3,5) and (4,9,–3) as end points of a diameter

Sol^n : Required equation of sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$\Rightarrow \quad (x-2)(x-4)+(y-3)(y-9)+(z-5)(z+3)=0$$

$$\Rightarrow \quad x^2+y^2+z^2-6x-12y-2z+20=0$$

General form of the equation of a sphere :

We have equation of the sphere with centre (a, b, c) and radius r is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$$\Rightarrow \quad x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2-r^2=0$$

If we put $-a=u, -b=v, -c=w$ and $a^2+b^2+c^2-r^2=d$, then we get

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

which is considered as the general equation of a sphere having centre $(-u, -v, -w)$ and radius $\sqrt{(u^2+v^2+w^2-d)}$

Note : 1. The general equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ suggests that any equation of the second degree will represent a sphere provided that

(i) the coefficint s of x^2, y^2 and z^2 are equal

(ii) there is no term involving the products xy, yz and zx

2. To find the equation of sphere passing through given four non - coplanar points, we assume the equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ and determine the values values of u, v, w, d by using four conitions.

Method to find centre and radius of sphere :

Let the equation of given sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Then, centre $\left(\dfrac{\text{coeff. of } x}{-2}, \dfrac{\text{coeff. of } y}{-2}, \dfrac{\text{coeff. of } z}{-2}\right)$

and radius $=\sqrt{\left(\dfrac{\text{coeff. of } x}{-2}\right)^2+\left(\dfrac{\text{coeff. of } y}{-2}\right)^2+\left(\dfrac{\text{coeff. of } z}{-2}\right)^2-d}$

provided the coefficien t of x^2, y^2 and z^2 are 1, if not make them 1 by dividing

Ex - 5 : Find centre and radius of the sphere $x^2 + y^2 + z^2 - 4x + 6y - 2z + 10 = 0$

Sol^n : Centre $\left(\frac{-4}{-2}, \frac{6}{-2}, \frac{-2}{-2}\right), i.e. (2,-3,1)$

Radius $= \sqrt{2^2 + (-3)^2 + 1 - 10} = \sqrt{4} = 2$

Ex - 6 : Find centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 6x + 2y - 4z - 3 = 0$

Sol^n : $2x^2 + 2y^2 + 2z^2 - 6x + 2y - 4z - 3 = 0$

$\Rightarrow x^2 + y^2 + z^2 - 3x + y - 2z - 3/2 = 0$

$\therefore$ Centre $\left(\frac{-3}{-2}, \frac{1}{-2}, \frac{-2}{-2}\right), i.e. \left(\frac{3}{2}, -\frac{1}{2}, 1\right)$

Radius $= \sqrt{\frac{9}{4} + \frac{1}{4} + 1 + \frac{3}{2}} = \sqrt{5}$

Ex - 7 : If one end point of the diameter of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 7 = 0$ is $(-1,2,4)$, find other end

Sol^n : Given sphere is $x^2 + y^2 + z^2 - 2x + 4y - 6z - 7 = 0$

Centre $\left(\frac{-2}{-2}, \frac{4}{-2}, \frac{-6}{-2}\right), i.e. (1,-2,3)$

Let other end point be (p, q, r)

Centre is midpoint of diameter of sphere

So, $\frac{-1+p}{2} = 1 \Rightarrow p = 3$

$\frac{2+q}{2} = -2 \Rightarrow q = -6$

$\frac{4+r}{2} = 3 \Rightarrow r = 2$

Hence other end point of diameter is $(3,-6,2)$

Ex - 8 : Find equation of sphere passing through t he point $(1,2,-3)$ and $(3,-1,2)$ and centre lying on Y - axis

Sol^n : Let centre of sphere be $C(0,k,0)$ ($\because$ on Y - axis , $x=0, z=0$)

Also sphere passes through points $P(1,2,-3)$ and $Q(3,-1,2)$

$\Rightarrow$ $CP = CQ$ ($\because$ radius)

$\Rightarrow$ $CP^2 = CQ^2$

$\Rightarrow$ $(1-0)^2+(2-k)^2+(-3-0)^2=(3-0)^2+(-1-k)^2+(2-0)^2$

$\Rightarrow$ $k=0$

$\therefore$ Centre$(0,0,0)$ and radius $= r = CP = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$

Hence equation of required sphere is

$$x^2+y^2+z^2=r^2$$

$\Rightarrow$ $x^2+y^2+z^2=14$

Ex - 9 : Find equation of sphere passing through origin and points $(a,0,0),(0,b,0),(0,0,c)$

Sol^n : Let the equation of required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \;.....(i)$$

Since the sphere passes through given four points, they satisfies the equation

for origin$(0,0,0)$,

$0+0+0+0+0+0+d=0$

$\Rightarrow$ $d=0$

for $(a,0,0)$,

$a^2+0+0+2ua+0+0+0=0$ ($\because d=0$)

$\Rightarrow$ $u=-a/2$

for $(0,b,0)$, $0+b^2+0+0+2vb+0+0=0$

$\Rightarrow$ $v=-b/2$

for $(0,0,c)$,

$0+0+c^2+0+0+2wc+0=0$

$\Rightarrow$ $w=-c/2$

Putting values of u,v,w,d in equation (i), we get

$x^2+y^2+z^2-ax-by-cz=0$ is the required equation of sphere

Assignment

1. Find equation of sphere whose centre is $(-2,3,1)$ and radius is 2
2. Find equation of sphere whose centre is $(4,2,1)$ and which passes through point $(-1,2,5)$
3. Find centre and radius of the sphere

   $(i)\ x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$

   $(ii)\ 3x^2 + 3y^2 + 3z^2 - 6x - 12y + 6z + 2 = 0$
4. Find equation of sphere joining points $(4,5,-6)$ and $(2,3,4)$ as end points of a diameter
5. Find equation of sphere whose centre at $(2,-1,4)$ and touches the plane $2x - y - 2z + 6 = 0$
6. Find equation of sphere passing through $(1,2,-3)$ and $(3,-1,2)$ and centre lying on X-axis
7. Find equation of sphere passing through origin and points $(1,0,0), (0,1,0)$ and $(0,0,1)$
8. For what value of a the equation $x^2 + y^2 - az^2 - 2x + 6y - 4z + 1 = 0$ represents a sphere