A rectangular array of mn numbers with m horizontal lines (rows) and n vertical lines (columns) is known as a matrix of order $m \times n$.

Ex: A= $\begin{bmatrix} 2 & 3 \\ a & b \\ -1 & 3 \end{bmatrix}$ it is a matrix of order 3×2 and contains 6 elements. i.e $3 \times 2 = 6$

General element of a matrix: If an element occurs in the ith row and jth column of a matrix then it is called (i,j)th element of the matrix.It is denoted by a_{ij}.

A matrix of order $m \times n$ *, generally written as* $A = [a_{ij}]_{m \times n}$

Let us consider a matrix of order 2×3 by general element.

A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Ex: construct a matrix of order 3×4 , whose elements are in the form of $a_{ij} = 2i - j$.

Types of a matrices:

Row matrix: A matrix having only one row is known as row matrix or Row vector.

A= $\begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$

Column matrix: A matrix having only one column is known as column matrix.

A= $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

Zero matrix or Null matrix: If all the elements of a matrix are zero then it is called null matrix.

Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Square matrix: If the number of rows and columns of a matrix are equal then it is called square matrix.i.e m=n

Ex: A = $\begin{bmatrix} 2 & -1 \\ 9 & 3 \end{bmatrix}$

Rectangular matrix: If the number of rows and columns of a matrix are not equal then it is called rectangular matrix. i.e $m \neq n$.

Ex:A= $\begin{bmatrix} -1 & 2 & 4 \\ 2 & 3 & -9 \end{bmatrix}$

Diagonal elements: The elements $a_{ij}, i = j$ in a square elements A are called diagonal elements.

Ex: A = $\begin{bmatrix} 2 & 3 & 0 \\ -2 & -3 & 1 \\ 4 & -6 & 4 \end{bmatrix}$ $here\ 2, -3\ and\ 4\ are\ diagonal\ elements.$

Diagonal matrix: A square matrix A is said to be a diagonal matrix if all the diagonal elements are present but non diagonal elements are zero, i.e $a_{ij} = 0, for\ all\ i \neq j.$

Ex: A = $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Scalar matrix: A square matrix A is said to be a scalar matrix if all the diagonal elements are equal but non diagonal elements are zero.

Ex: A = $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit matrix or Identity matrix: A square matrix is said to be an identity matrix if all the diagonal elements are unity(1) but non diagonal elements are zero. It is denoted by $I_n\ or\ I$.

i.e $a_{ij} = 0, for\ all\ i \neq j$ and $a_{ij} = 1, for\ all\ i = j.$

Ex: $I_3\ or\ I$= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex: $I_2\ or\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Upper triangular matrix: A square matrix is said to be an upper triangular matrix if all the elements below the main diagonal are zero. i.e $a_{ij} = 0, for\ all\ i > j.$

Ex: A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

Lower triangular matrix: A square matrix is said to be a lower triangular matrix if all the elements above the main diagonal are zero. $i.e\ a_{ij} = 0, for\ all\ i < j$

Ex: A = $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & -2 \end{bmatrix}$

Singular matrix: A square matrix A is said to be a singular matrix if det.A=0 or $|A| = 0$.

Ex: let A=$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \ hence\ A\ is\ a\ singular\ matrix.$$

Non singular matrix: A square matrix A is said to be a nonsingular matrix if or det $A \neq 0\ or\ |A| \neq 0$.

Ex: let A=$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 = -4\ \ hence\ A\ is\ a\ nonsingular\ matrix.$$

Comparable Matrix: Two matrices A and B are said to be comparable if they have same order.

Ex: $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ here A and B are comparable.

Equal Matrix: Two matrices A and B are said to be equal (i.e A=B) if they have same order and their corresponding elements are equal.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}$ So A = B iff $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2, e_1 = e_2, f_1 = f_2$

Ex: Find the value of x and y if $\begin{bmatrix} 2x - y & -1 \\ 2 & x + y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix}$

Ans: Here 2x-y=3 and x+y=6 solve the above two equations x=3 and y=3.

Scalar multiplication of a matrix: If A be a matrix and k be a scalar then the scalar multiplication of a matrix kA will be obtained multiplying each element of A by k.$i.e\ A = [a_{ij}] \Rightarrow kA = [ka_{ij}]$

Ex: let A=$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 10 & 12 \end{bmatrix}$

Matrix addition:

If A and B are two matrices of order $m \times n$ then addition A+B will be obtained by adding the corresponding elements of A and B. The order of A+B is $m \times n$.

Ex: Find A+B if $A = \begin{bmatrix} 2 & -1 & 3 \\ -3 & -2 & 4 \end{bmatrix}\ and\ B = \begin{bmatrix} 5 & 4 & 3 \\ 3 & -6 & 1 \end{bmatrix}$

Ans: $A+B=\begin{bmatrix}2 & -1 & 3\\ -3 & -2 & 4\end{bmatrix}+\begin{bmatrix}5 & 4 & 3\\ 3 & -6 & 1\end{bmatrix}=\begin{bmatrix}2+5 & -1+4 & 3+3\\ -3+3 & -2-6 & 4+1\end{bmatrix}=\begin{bmatrix}7 & 3 & 6\\ 0 & -8 & 5\end{bmatrix}$

Similarly $A-B=\begin{bmatrix}2 & -1 & 3\\ -3 & -2 & 4\end{bmatrix}-\begin{bmatrix}5 & 4 & 3\\ 3 & -6 & 1\end{bmatrix}=\begin{bmatrix}2-5 & -1-4 & 3-3\\ -3-3 & -2+6 & 4-1\end{bmatrix}=$
$\begin{bmatrix}-3 & -5 & 0\\ -6 & 4 & 3\end{bmatrix}$

Note:

1) $A+B=B+A$

2) $A-B\neq B-A$

3) $k(A+B)=kA+kB$

4) $(\alpha+\beta)A=\alpha A+\beta A$

5) $\alpha\beta A=\alpha(\beta A)=\beta(\alpha A)$

6) A+(B+C)=(A+B)+C

7) **Existence of Additive Identity**: A+O=O+A=A(Here O be a Null matrix)

8) **Existence of Additive Inverse**: A+(-A)=O=(-A)+A

9) **Cancellation law**: A+B=A+C$\Rightarrow B=C(left\ cancellation\ law)$

$$B+A=C+A\Rightarrow B=C(right\ cancellation\ law)$$

Matrix Multiplication:

Existence of the product of two matrices: The product of two matrices A and B are said to be exist $(i.e\ AB\ exist)$ if the number of columns in the matrix A is equal to the number of rows in the matrix B.

Product of matrices:

$If\ \ A=[a_{ij}]_{m\times n}\ and\ B=[b_{jk}]_{n\times p}\ then\ AB\ is\ a\ matrix\ of\ order\ m\times p$, Which is defined as $AB=[c_{ik}]_{m\times p}, where\ \ c_{ik}=a_{i1}b_{1k}+a_{i2}b_{2k}+------+a_{in}b_{nk}=\sum_{j=1}^{n}a_{ij}b_{jk}$

Thus $(i,k)th\ element\ of\ AB=Sum\ of\ the\ product\ of\ the\ corresponding\ elementsof$

$ith\ row\ of\ A\ and\ kth\ column\ of\ B$

Properties of product of two matrices:

1.Matrix multiplication is not commutative in general $i.e\ AB \neq BA$

2.Matrix multiplication is associative $i.e\ A(BC) = (AB)C$

3.Matrix multiplication is distributive over addition $i.e\ A(B+C) = AB + AC$

4.$A.A = A^2$

5.$A.I = IA = A$

6.$I.I.I ------I(ntimes) = I$

Transpose of a matrix:

If A be a matrix of order $m \times n$ then the transpose of the matrix will be obtained by interchanging the rows and columns. It is denoted by A^T and the order of A^T $is\ n \times m$.

Ex: let $A = \begin{bmatrix} 2 & -3 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 3 \\ -3 & 2 \\ 4 & 5 \end{bmatrix}$

Properties:

1)$(A^T)^T$=A

2)$(A+B)^T = A^T + B^T$

3)If A be a matrix and k is a scalar $(kA)^T = k\ A^T$

4)$(AB)^T = B^T A^T$

5) $(ABC)^T = C^T B^T A^T$

Symmetric Matrix:A square matrix A is said to be symmetric matrix if $A^T = A, i.e\ a_{ij} = a_{ji}\ for\ all\ i\ and\ j$

Ex: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.$A = \begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}$

Skew-symmetric matrix:A square matrix A is said to be skew-symmetric matrix if $A^T = -A, i.e\ a_{ij} = -a_{ji}\ for\ all\ i\ and\ j$

Ex: $A = \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$.

Properties:

1)$A + A^T$ *is a symmetric matrix but* $A - A^T$ *is a skew − symmetric matrix*.

2)AA^T *and* A^TA *are symmetric matrix*

3)Every square matrix A can be expressed as the sum of symmetric and skew-symmetric matrix.

$$i.e\ A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\ Here\ \frac{1}{2}(A + A^T)\ is\ a\ symmetric\ matrix\ and$$

$$\frac{1}{2}(A - A^T) is\ a\ skew - symmetric\ matrix$$

Note: If A and B are symmetric matrix and AB=BA then AB is a symmetric matrix.

Note:If A is a symmetric matrix then A^n *is a symmetric matrix for all + ve integer*.

Note:A matrix which is both symmetric as well as skew symmetric matrix is a null matrix.

Adjoint of a matrix:

Adjoint of a square matrix A is defined as the transpose of the cofactor matrix A.

It is denoted adj.A.

$$i.e\ adj.A = (cofactor\ matrix\ of\ A)^T = \left[C_{ij}\right]^T, Where\ C_{ij}\ is\ cofactor\ of\ a_{ij} in\ A$$

Note: If A be a square matrix of order n then A(adjA)=$|A|I_n$, *I be an Identity matrix*

Note: If A be a non–singular matrix of order n then $|adjA| = |A|^{n-1}$

Note: If A and B are non-singular square matrix of same order then adjAB=adjBadjA

Note: If A is a non-singular matrix then adj(adjA)= $|A|^{n-2}A$

Inverse of a matrix:

A non- singular square matrix A of order n is said to be invertible (i.e A^{-1} exists) if there exist a non singular square matrix B of order n, such that $AB = BA = I, so\ A^{-1} = B\ or\ B^{-1} = A$.

Formula finding A^{-1}:

If A be non singular square matrix then A^{-1} is defined as $A^{-1} = \frac{adj.A}{|A|}$

Note: If A is invertible matrix then $(A^{-1})^{-1} = A$

Note:$(AB)^{-1} = B^{-1}A^{-1}$

Note: If A is invertible square matrix then $A^T\ is\ invertible\ i.e\ (A^T)^{-1} = (A^{-1})^T$

Note: If A is an invertible square matrix then adj$A^T = (adjA)^T$

Note: Adjoint of a symmetric matrix is also a symmetric matrix.

$$(adjA)^T = adjA$$

Note: If A is a non-singular matrix then $|A^{-1}| = |A|^{-1}$

Solution for system of linear equations:(Solve by matrix method)

Let us consider three linear equations

$a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2\ and\ a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Now the above equation can be written as $AX = B \Rightarrow X = A^{-1}B$

Which is known as matrix method.