# Some definitions Related to limit

Constant: A quantity which does not change its value under different mathematical operations of a problem is called constant.

$Ex: 5, \pi, e, \sqrt{3}\, etc.$ are all constants.

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Variable : A quantity which changes its value under different mathematical operations of a problem is called a variable.

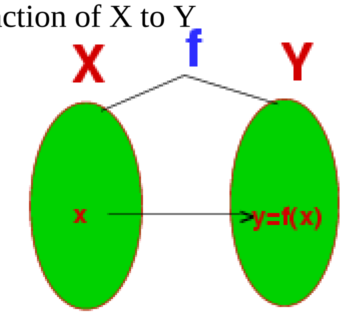
Ex:The equation of a straight line $y = mx + c$. Here x&y are variables.

Indepedent variable :A variable(x), which can assumes any arbitrary value from a set is called an independent variable.

Dependent variable: A variable (y), whose value depends on the value of another independent variable (x) is called dependent variable.

Ex:Area of a circle is $A = \pi r^2$.Here A is the dependent variable & r is an independent variable.

Function or Mapping : Let X and Y be any two non empty sets and there be a correspondence f between the elements of X and Ysuch that for every element $x \in X$, there exists a unique and definite element $y \in Y$, which is written as $y = f(x)$, then we say that f is a function of X to Y ,written as $f : X \to Y$.




Note-1:Every element of X will be mapped by f.

2:The element of y $\in Y$ is called the image of x under f

Meaning of f(a):The value of the function $y = f(x)$ at $x = a$ denoted by f(a). It is obtained by putting a in place of x in f(x).

Domain and range of a function : Let $f : X \to Y$ be a mapping from a set X to the set Y.

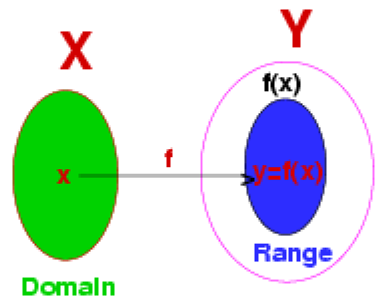
Domain: The domain of $f = X$

Range: The range of $f = f(x) = \{ f(x) : f(x) \in Y, x \in X \}$ = set of all image points inY.

Note:1.If $x \in X$ , then $f(x) \in f(X)$

2.$f(X) \subseteq Y$ i.e Range of $f \subseteq y$

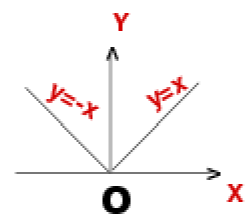
3.$Y$ is called the co-domain of $f$




Real valued function : If $f : X \rightarrow Y$ be a function from a set $X$ to a set $Y$ where $X, Y \subseteq R$, then $f$ is a real valued function.

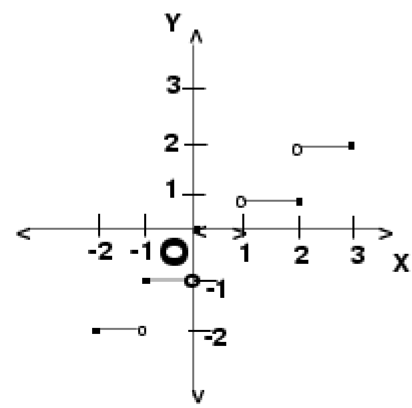
Ex: $y = f(x) = \dfrac{x+1}{x^2+1}, x \in R$

Modulus function:The modulus of a real number $x$ denoted by is $|x|$is its magnitude or numerical value taken with a positive sign.




i.e $|x| = \begin{cases} x & if\ x > 0 \\ -x & if\ x < 0 \\ 0 & if\ x = 0 \end{cases}$

Greatest integer function:The greatest integer function of $x$ is denoted by $[x]$ is the greatest integer less than or equal to $x$ $i.e$ $[x] \leq x.$




$y = [x] = \begin{cases} -1, -1 \leq x < 0 \\ 0, 0 \leq x < 1 \\ 1, 1 \leq x < 2 \\ 2, 1 \leq x < 2........ and\ so\ on. \end{cases}$

Ex: $|3| = 3, |\text{-}3| = 3, |0| = 0,$

Ex: $[2] = 2, [\text{-}3] = -3, [0] = 0, [2.98] = 2, [\text{-}2.98] = -3, [0.5] = 0, [\text{-}0.5] = -1,$

$[2\text{-h}] = 1, [2 + \text{h}] = 2$ where h is a small $+ve$ quantity.

Logarithm function: If $a > 0$ and $a \neq 1$ then $y = \log_a x$ is called a logarithm function with base $a$, for all $x > 0$.

Note: 1. $\log_a a = 1$ , $\log_a 1 = 0$ , $\log_e x = \ln x$

2. $\log_a 0 = \begin{cases} -\infty, if\ a > 0 \\ +\infty, if\ 0 < a < 1 \end{cases}$

$Ex: y = \log_e(x+1)$ , $y = \log_3(x^2+2)$

Exponential function: The function $y = a^x\ or\ y = e^x$ is called an exponential function, for all $x \in R$.

Ex: $y = 2^x, y = x^{\sin x}$ , $y = e^{\cos x}$

Note:1: $e^{\ln x} = x = e^{\log_e x}$, $a^x = e^{x\log_e x}$ , $a^{\log_a x} = x, (a > 0, a \neq 1)$

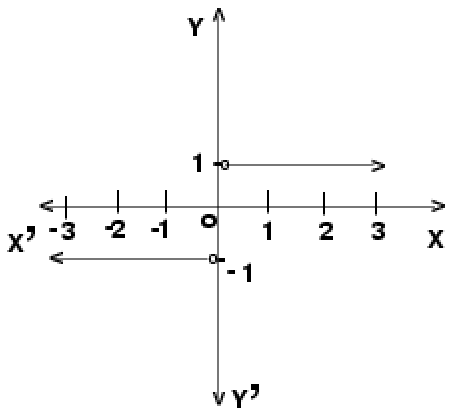
2 : $\log_e mn = \log_e m + \log_e n$

3 : $\log_e \frac{m}{n} = \log_e m - \log_e n$

4 : $\log_e m^n = n\log_e m$

Signum function: The signum function is defined as

$$f(x) = \begin{cases} \frac{|x|}{x}, if\ x \neq 0 \\ 0, if\ x = 0 \end{cases}$$




**LIMIT OF A FUNCTION** **The limit of a function f(x) is defined as the value of a function at some indicated points.**

The limit or limiting value of a function is the fundamental concept of calculus.

Definition: If '$l$' be the limiting value of a function $f(x)$ at a point $x = a$ then which is written as

$$\lim_{x \to a} f(x) = l \text{ OR } \underset{x \to a}{lt} \left( \begin{array}{c} \text{which is read as the value of } f(x) \text{ tends to } l \text{ as } x \text{ tends to } a \\ \text{i.e } f(x) \to l \text{ as } x \to a \end{array} \right)$$

$Note: x \to a$ means $x$ close to $a$ but $x \neq a$

Limit or limiting value of a function: (Meaning of $\lim_{x \to a}$)

If y = $f(x)$ be a given function of x then for every value of x we obtain a definite and unique value of. But sometimes for a particular value of x, the value of f $(x)$ is not determinate.

For example Let $y = f(x) = \frac{x^2 - 9}{x - 3} \quad at\ x = 3$

$f(3) = \frac{3^2 - 9}{3 - 3} = \frac{0}{0}$ which is not determinate.

In maximum case we obtain the limiting value of the function at a point where it is indeterminate.

## Algebra of limits

1. $\lim_{x \to a}\left(f(x) + g(x)\right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}\left(f(x) \times g(x)\right) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
3. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\ where \lim_{x \to a} g(x) \neq 0$
4. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$ Where k is a constant.
5. $\lim_{x \to a} k = k$ Where k is a constant

## LIMIT FORMULAS

$F - 1. For\ n\ be\ any\ rational$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$F - 2. \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$F - 3. \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a\ or\ \log_e a\ (a > 0)$$

$$F - 4. \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1\ or \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1\ or \lim_{x \to 0} \frac{x}{\ln(1+x)} = 1$$

$$F - 5. \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

$$F - 6. \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$F - 7. \lim_{x \to 0} (1 + \lambda x)^{\frac{1}{x}} = e^{\lambda}$$

$F-8.\lim_{x\to 0}\frac{\sin x}{x}=1 \, or \lim_{x\to 0}\frac{x}{\sin x}=1$

$F-9.\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1 \, or \lim_{x\to 0}\frac{x}{\sin^{-1}x}=1$

$F-10.\lim_{x\to 0}\frac{\tan x}{x}=1 \, or \lim_{x\to 0}\frac{x}{\tan x}=1$

$F-11.\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1 \, or \lim_{x\to 0}\frac{x}{\tan^{-1}x}=1$

$F-12.\lim_{x\to 0}\sin x=0, \lim_{x\to 0}\cos x=1$

$F-13.\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$

$F-14.\lim_{x\to 0}\frac{(1-x)^n-1}{x}=-n$

## Rules for finding $\lim_{x\to a} f(x)$

$Rule-1. Put\ x=a\ If\ f(a)\ gives\ a\ finite\ value$

$Ex:\lim_{x\to 2}\left(x^2+3x+1\right)$

$Ans:\lim_{x\to 2}\left(x^2+3x+1\right)=2^2+3\times 2+1=11$

$Ex:\lim_{x\to 3}\left(x^2+3x\right)\left(x-2\right)$

$Ans:\lim_{x\to 3}\left(x^2+3x\right)\left(x-2\right)=\lim_{x\to 3}\left(x^2+3x\right)\times\lim_{x\to 3}\left(x-2\right)=\left(3^2+3\times 2\right)\times\left(3-2\right)=15\times 1=15$

$Ex:\lim_{x\to 1}\frac{x^2-2x+5}{x+3}=\frac{\lim_{x\to 1}\left(x^2-2x+5\right)}{\lim_{x\to 1}\left(x+3\right)}=\frac{1^2-2\times 1+5}{1+3}=\frac{4}{4}=1$

$Rule-2.\left(\frac{0}{0}\ form\right)$

$If\ f(x)\ be\ a\ rational\ function\ then\ factorize\ both\ numerator(N^r)\ \&\ denominator(D^r).cancel\ the\ common\ factor\ and\ put\ x=a.$

$Ex:\lim_{x\to 2}\frac{x^2-4}{x-2}$

$Ans:\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\left(x-2\right)\left(x+2\right)}{x-2}=\lim_{x\to 2}\left(x+2\right)=2+2=4$

$Ex:\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$

$Ans:\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{x-1}=\lim\limits_{x\to 1}(x^2+x+1)=1+1+1=3$

$Ex:\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x-2}$

$Ans:\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x-2}=\lim\limits_{x\to 2}\dfrac{x^2-3x-2x+6}{x-2}=\lim\limits_{x\to 2}\dfrac{x(x-3)-2(x-3)}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(x-3)}{x-2}$

$=\lim\limits_{x\to 2}(x-3)=2-3=-1$

Rule-3.If the given function contains a surd, then multiply a conjugate surd in both $N^r$ & $D^r$ ,after simplification put $x=a$.

$Ex:\lim\limits_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

$Ans:\lim\limits_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{x+2}-\sqrt{2}\right)\left(\sqrt{x+2}+\sqrt{2}\right)}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim\limits_{x\to 0}\dfrac{(x+2-2)}{x\left(\sqrt{x+2}+\sqrt{2}\right)}$

$=\lim\limits_{x\to 0}\dfrac{x}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim\limits_{x\to 0}\dfrac{1}{\left(\sqrt{x+2}+\sqrt{2}\right)}=\dfrac{1}{\sqrt{2}+\sqrt{2}}(put\ x=0)$

$=\dfrac{1}{2\sqrt{2}}$

$Ex:\lim\limits_{x\to 0}\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}$

$Ans:\lim\limits_{x\to 0}\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}=\lim\limits_{x\to 0}\dfrac{(\sqrt{2-x}-\sqrt{2+x})\left(\sqrt{2-x}+\sqrt{2+x}\right)}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}$

$=\lim\limits_{x\to 0}\dfrac{(2-x)-(2+x)}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}=\lim\limits_{x\to 0}\dfrac{2-x-2-x}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}=\lim\limits_{x\to 0}\dfrac{-2x}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}=\dfrac{-2}{\left(\sqrt{2-x}+\sqrt{2+x}\right)}$

$=\dfrac{-2}{\sqrt{2}+\sqrt{2}}=\dfrac{-2}{2\sqrt{2}}=\dfrac{-1}{\sqrt{2}}$

$Ex-\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ . $Ex-\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+4}-2}$ . $Ex-\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}$ . $Ex-\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{x}$

$Ex-\lim\limits_{x\to a}\dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$

$$Ex:\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$$

$$Ans:\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x^2-\sqrt{x})\times(x^2+\sqrt{x})\times(\sqrt{x}+1)}{(\sqrt{x}-1)\times(\sqrt{x}+1)\times(x^2+\sqrt{x})}(double\,conjugate\,multiplication)$$

$$=\lim_{x\to 1}\frac{(x^4-x)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}=\lim_{x\to 1}\frac{x(x^3-1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}=\lim_{x\to 1}\frac{x(x-1)(x^2+x+1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}$$

$$=\lim_{x\to 1}\frac{x(x^2+x+1)(\sqrt{x}+1)}{(x^2+\sqrt{x})}=\frac{1\times 3\times 2}{2}=3$$

$$Ex-\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+3}-\sqrt{3}}.\ Ex-\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+4}-2}.$$

$$Formula:\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$Ex:\lim_{x\to 2}\frac{x^9-2^9}{x-2}$$

$$Ans:\lim_{x\to 2}\frac{x^9-2^9}{x-2}=9\times 2^8$$

$$Ex:\lim_{x\to 2}\frac{x^5-32}{x-2}$$

$$Ans:\lim_{x\to 2}\frac{x^5-32}{x-2}=\lim_{x\to 2}\frac{x^5-2^5}{x-2}=5\times 2^4=32$$

$$Ex:\lim_{x\to 2}\frac{x^5-32}{x^3-8}$$

$$Ans:\lim_{x\to 2}\frac{x^5-32}{x^3-8}=\frac{\lim_{x\to 2}\frac{x^5-2^5}{x-2}}{\lim_{x\to 2}\frac{x^3-2^3}{x-2}}=\frac{5\times 2^4}{3\times 2^2}=\frac{32}{12}=\frac{8}{3}$$

$$Ex:\lim_{x\to 1}\frac{x^n-1}{x^m-1}$$

$$Ans:\lim_{x\to 1}\frac{x^n-1}{x^m-1}=\frac{\lim_{x\to 1}\frac{x^n-1^n}{x-1}}{\lim_{x\to 1}\frac{x^m-1^m}{x-1}}=\frac{n\times 1^{n-1}}{m\times 1^{m-1}}=\frac{n}{m}$$

$$Ex:\lim_{x\to 5}\frac{\sqrt{x}-\sqrt{5}}{x-5},\ Ex:\lim_{x\to 3}\frac{x^4-81}{x-3},\ Ex:\lim_{x\to 3}\frac{x^{\frac{1}{n}}-1}{x^{\frac{1}{m}}-1},\ Ex:\lim_{x\to b}\frac{x^8-b^8}{x^3-b^3}$$

$$Formula:\lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$Ex:\lim_{x\to 0}\frac{e^{ax}-1}{x}$$

$$Ans:\lim_{x\to 0}\frac{e^{ax}-1}{x}=\lim_{x\to 0}\frac{e^{ax}-1}{ax}\times a=1\times a=a$$

$$Ex:\lim_{x\to 0}\frac{e^{-3x}-1}{5x}$$

$$Ans:\lim_{x\to 0}\frac{e^{-3x}-1}{5x}=\frac{1}{5}\lim_{x\to 0}\frac{e^{-3x}-1}{x}=\frac{1}{5}\lim_{x\to 0}\frac{e^{-3x}-1}{-3x}\times -3=\frac{1}{5}\times 1\times(-3)=\frac{-3}{5}$$

$$Ex:\lim_{x\to 0}\frac{e^{-3x}-1}{e^{5x}-1}$$

$$Ans:\lim_{x\to 0}\frac{e^{-3x}-1}{e^{5x}-1}=\frac{\lim_{x\to 0}\frac{e^{-3x}-1}{x}}{\lim_{x\to 0}\frac{e^{5x}-1}{x}}=\frac{\lim_{x\to 0}\frac{e^{-3x}-1}{-3x}\times -3}{\lim_{x\to 0}\frac{e^{5x}-1}{5x}\times 5}=\frac{1\times(-3)}{1(5)}=\frac{-3}{5}$$

$$Ex:\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{x}$$

$$Ans:\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{x}=\lim_{x\to 0}\frac{e^{ax}-1+1-e^{-ax}}{x}=\lim_{x\to 0}\frac{(e^{ax}-1)-(e^{-ax}-1)}{x}$$

$$=\lim_{x\to 0}\frac{e^{ax}-1}{x}-\lim_{x\to 0}\frac{e^{-ax}-1}{x}=\lim_{x\to 0}\frac{e^{ax}-1}{ax}\times a-\lim_{x\to 0}\frac{e^{-ax}-1}{-ax}\times(-a)=1\times a-1\times(-a)=2a$$

$$Ex:\lim_{x\to 0}\frac{e^{-3x}-e^{4x}}{e^{5x}-e^{-2x}}$$

$$Ans:\lim_{x\to 0}\frac{e^{-3x}-e^{4x}}{e^{5x}-e^{-2x}}=\frac{\lim_{x\to 0}\frac{e^{-3x}-e^{4x}}{x}}{\lim_{x\to 0}\frac{e^{5x}-e^{-2x}}{x}}=\frac{\lim_{x\to 0}\frac{e^{-3x}-1+1-e^{4x}}{x}}{\lim_{x\to 0}\frac{e^{5x}-1+1-e^{-2x}}{x}}=\frac{\lim_{x\to 0}\frac{(e^{-3x}-1)-(e^{4x}-1)}{x}}{\lim_{x\to 0}\frac{(e^{5x}-1)-(e^{-2x}-1)}{x}}$$

$$=\frac{\lim_{x\to 0}\frac{e^{-3x}-1}{x}-\lim_{x\to 0}\frac{(e^{4x}-1)}{x}}{\lim_{x\to 0}\frac{(e^{5x}-1)}{x}-\lim_{x\to 0}\frac{(e^{-2x}-1)}{x}}=\frac{\lim_{x\to 0}\frac{e^{-3x}-1}{-3x}\times(-3)-\lim_{x\to 0}\frac{(e^{4x}-1)}{4x}\times 4}{\lim_{x\to 0}\frac{(e^{5x}-1)}{5x}\times 5-\lim_{x\to 0}\frac{(e^{-2x}-1)}{-2x}\times(-2)}=\frac{-3-4}{5-(-2)}=\frac{-7}{7}=-1$$

$$Ex:\lim_{x\to 0}\frac{e^{3x}-1}{x},\ Ex:\lim_{x\to 0}\frac{e^{3x}-1}{e^{-3x}-1},\ Ex:\lim_{x\to 0}\frac{e^{3x}-e^{5x}}{x},\ Ex:\lim_{x\to 0}\frac{e^{ax}-e^{cx}}{e^{bx}-e^{dx}}$$

$$Formula: \lim_{x\to 0}\frac{a^x-1}{x}=\ln a \; or \; \log_e a (a>0)$$

$$Ex: \lim_{x\to 0}\frac{3^x-1}{x}$$

$$Ans: \lim_{x\to 0}\frac{3^x-1}{x}=\ln 3 \; or \; \log_e 3$$

$$Ex: \lim_{x\to 0}\frac{a^{2x}-1}{x}$$

$$Ans: \lim_{x\to 0}\frac{a^{2x}-1}{x}=\lim_{x\to 0}\frac{a^{2x}-1}{2x}\times 2=\ln a\times 2=2\ln a=\ln a^2$$

$$Ex: \lim_{x\to 0}\frac{b^{3x}-1}{5x}$$

$$Ans: \lim_{x\to 0}\frac{b^{3x}-1}{5x}=\frac{1}{5}\lim_{x\to 0}\frac{b^{3x}-1}{x}=\frac{1}{5}\lim_{x\to 0}\frac{b^{3x}-1}{3x}\times 3=\frac{1}{5}\times\ln b\times 3=\frac{3}{5}\times\ln b$$

$$Ex: \lim_{x\to 0}\frac{a^x-1}{b^x-1}$$

$$Ans: \lim_{x\to 0}\frac{a^x-1}{b^x-1}=\frac{\lim_{x\to 0}\frac{a^x-1}{x}}{\lim_{x\to 0}\frac{b^x-1}{x}}=\frac{\ln a}{\ln b}=\log_b a$$

$$Ex: \lim_{x\to 0}\frac{a^x-b^x}{x}$$

$$Ans: \lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{a^x-1+1-b^x}{x}=\lim_{x\to 0}\frac{(a^x-1)-(b^x-1)}{x}=$$

$$=\lim_{x\to 0}\frac{a^x-1}{x}-\lim_{x\to 0}\frac{b^x-1}{x}=\ln a-\ln b=\ln\frac{a}{b}$$

$$Ex: \lim_{x\to 0}\frac{a^x-b^x}{c^x-d^x}$$

$$Ans: \lim_{x\to 0}\frac{a^x-b^x}{c^x-d^x}=\frac{\lim_{x\to 0}\frac{a^x-b^x}{x}}{\lim_{x\to 0}\frac{c^x-d^x}{x}}=\frac{\lim_{x\to 0}\frac{a^x-1+1-b^x}{x}}{\lim_{x\to 0}\frac{c^x-1+1-d^x}{x}}=\frac{\lim_{x\to 0}\frac{(a^x-1)-(b^x-1)}{x}}{\lim_{x\to 0}\frac{(c^x-1)-(d^x-1)}{x}}$$

$$=\frac{\lim_{x\to 0}\frac{a^x-1}{x}-\lim_{x\to 0}\frac{b^x-1}{x}}{\lim_{x\to 0}\frac{c^x-1}{x}-\lim_{x\to 0}\frac{d^x-1}{x}}=\frac{\ln a-\ln b}{\ln c-\ln d}=\frac{\ln\frac{a}{b}}{\ln\frac{c}{d}}$$

$Ex:\lim_{x\to0}\frac{3^x-1}{5^x-1}, Ex:\lim_{x\to0}\frac{3^x-5^x}{x}$ , $Ex:\lim_{x\to0}\frac{3^x-4^x}{5^x-7^x}$

$Formula:\lim_{x\to0}\frac{\log_e(1+x)}{x}=1$

$Ex:\lim_{x\to0}\frac{\log_e(1+2x)}{x}$

$Ans:\lim_{x\to0}\frac{\log_e(1+2x)}{x}=\lim_{x\to0}\frac{\log_e(1+2x)}{2x}\times2=1\times2=2$

$Ex:\lim_{x\to0}\frac{\log_e(1+2x)}{5x}$

$Ans:\lim_{x\to0}\frac{\log_e(1+2x)}{5x}=\frac{1}{5}\lim_{x\to0}\frac{\log_e(1+2x)}{x}=\frac{1}{5}\lim_{x\to0}\frac{\log_e(1+2x)}{2x}\times2=\frac{1}{5}\times1\times2=\frac{2}{5}$

$Ex:\lim_{x\to0}\frac{\log_e(1-3x)}{x}$

$Ans:\lim_{x\to0}\frac{\log_e(1-3x)}{x}=\lim_{x\to0}\frac{\log_e\{1+(-3)x\}}{(-3)x}\times(-3)=1\times(-3)=-3$

$Ex:\lim_{x\to0}\frac{\log_e\left(1+\frac{x}{2}\right)}{x}$, $Ex:\lim_{x\to0}\frac{\log_e(1-2x)}{5x}$

$Formula:\lim_{x\to0}(1+x)^{\frac{1}{x}}=e$

$Ex:\lim_{x\to0}(1+3x)^{\frac{1}{x}}$

$Ans:\lim_{x\to0}(1+3x)^{\frac{1}{x}}=\lim_{x\to0}(1+3x)^{\frac{1}{3x}\times3}=\left(\lim_{x\to0}(1+3x)^{\frac{1}{3x}}\right)^3=e^3$

$Ex:\lim_{x\to0}(1+3x)^{\frac{2}{5x}}$

$Ans:\lim_{x\to0}(1+3x)^{\frac{2}{5x}}=\lim_{x\to0}(1+3x)^{\frac{1}{x}\times\frac{2}{5}}=\lim_{x\to0}(1+3x)^{\frac{1}{3x}\times3\times\frac{2}{5}}=\lim_{x\to0}(1+3x)^{\frac{1}{3x}\times\frac{6}{5}}$

$=\left(\lim_{x\to0}(1+3x)^{\frac{1}{3x}}\right)^{\frac{6}{5}}=e^{\frac{6}{5}}$

$Ex:\lim_{x\to0}(1+2x)^{\frac{1}{x}}, Ex:\lim_{x\to0}(1-2x)^{\frac{1}{5x}}$

$Formula: \lim_{x \to 0} \frac{\sin x}{x} = 1$

$Ex: \lim_{x \to 0} \frac{\sin 3x}{x}$

$Ans: \lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \times 3 = 1 \times 3 = 3$

$Ex: \lim_{x \to 0} \frac{\sin 3x}{5x}$

$Ans: \lim_{x \to 0} \frac{\sin 3x}{5x} = \frac{1}{5} \lim_{x \to 0} \frac{\sin 3x}{x} = \frac{1}{5} \lim_{x \to 0} \frac{\sin 3x}{3x} \times 3 = \frac{1}{5} \times 3 = \frac{3}{5}$

$Ex: \lim_{x \to 0} \frac{\sin \frac{x}{2}}{x}$

$Ans: \lim_{x \to 0} \frac{\sin \frac{x}{2}}{x} = \lim_{x \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} = 1 \times \frac{1}{2}$

$Ex: \lim_{x \to 0} \frac{\sin^2 x}{x^2}$

$Ans: \lim_{x \to 0} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2 = 1^2 = 1$

$Ex: \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{x^2}$

$Ans: \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{x^2} = \lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{x} \right)^2 = \lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 = \lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} = 1^2 \times \frac{1}{4} = \frac{1}{4}$

$Ex: \lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$Ans: \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2} = 2\lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{x} \right)^2 = 2\lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \right)^2 = 2\lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4}$

$= 2 \times 1^2 \times \frac{1}{4} = \frac{1}{2}$

*OR*

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{(1-\cos x)\times(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{(1-\cos^2 x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{1}{(1+\cos x)}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times\frac{1}{2}=1\times\frac{1}{2}=\frac{1}{2}$$

$$Ex:\lim_{x\to 0}\frac{\tan ax}{x}$$

$$Ans:\lim_{x\to 0}\frac{\tan ax}{x}=\lim_{x\to 0}\frac{\tan ax}{ax}\times a=1\times a=a\left(\because \lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$Ex:\lim_{x\to 0}\frac{\sin^{-1}4x}{x}$$

$$Ans:=\lim_{x\to 0}\frac{\sin^{-1}4x}{x}=\lim_{x\to 0}\frac{\sin^{-1}4x}{4x}\times 4=1\times 4=4\left(\because \lim_{x\to 0}\frac{\sin^{-1}x}{x}=1\right)$$

$$Ex:\lim_{x\to 0}\frac{\tan^{-1}bx}{x}$$

$$Ans:\lim_{x\to 0}\frac{\tan^{-1}bx}{x}=\lim_{x\to 0}\frac{\tan^{-1}bx}{bx}\times b=1\times b=b\left(\because \lim_{x\to 0}\frac{\tan^{-1}x}{x}=1\right)$$

$$Ex:\lim_{x\to 0}\frac{x}{\sin 2x}$$

$$Ans:\lim_{x\to 0}\frac{x}{\sin 2x}=\lim_{x\to 0}\frac{2x}{\sin 2x}\times\frac{1}{2}=1\times\frac{1}{2}=\frac{1}{2}\left(\because \lim_{x\to 0}\frac{x}{\sin x}=1\right)$$

$$Ex:\lim_{x\to 0}\frac{\sin mx}{\sin nx}$$

$$Ans:\lim_{x\to 0}\frac{\sin mx}{\sin nx}=\frac{\lim_{x\to 0}\frac{\sin mx}{x}}{\lim_{x\to 0}\frac{\sin nx}{x}}=\frac{\lim_{x\to 0}\frac{\sin mx}{mx}\times m}{\lim_{x\to 0}\frac{\sin nx}{nx}\times n}=\frac{1\times m}{1\times n}=\frac{m}{n}$$

$$Ex:\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ Ex:\lim_{x\to 0}\frac{\tan 3x}{\tan 5x},\ Ex:Ex:\lim_{x\to 0}\frac{\tan 3x}{5x},\ Ex:\lim_{x\to 0}\frac{\sin 2x}{7x}$$

## Some problems by using substitution:

$$Ex:\lim_{h\to 0}\frac{(x+h)^5-x^5}{h}$$

$$Ans: \lim_{h\to 0}\frac{(x+h)^5-x^5}{h}\begin{cases}Let\ x+h=y\Rightarrow h=y-x\\ As\ h\to 0\Rightarrow y\to x\end{cases}$$

$$=\lim_{y\to x}\frac{y^5-x^5}{y-x}=5x^{5-1}=5x^4$$

$$Ex: \lim_{x\to 1}\frac{\ln x}{x-1}$$

$$Ans: \lim_{x\to 1}\frac{\ln x}{x-1}\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to 1\Rightarrow y\to 0\end{cases}$$

$$=\lim_{y\to 0}\frac{\ln(1+y)}{y}=1$$

$$Ex: \lim_{x\to 1}\frac{\ln(2x-1)}{x-1}$$

$$Ans: \lim_{x\to 1}\frac{\ln(2x-1)}{x-1}\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to 1\Rightarrow y\to 0\end{cases}$$

$$=\lim_{y\to 0}\frac{\ln\left(2(1+y)-1\right)}{y}=\lim_{y\to 0}\frac{\ln\left(2+2y-1\right)}{y}=\lim_{y\to 0}\frac{\ln(1+2y)}{2y}\times 2=1\times 2=2$$

$$Ex-: \lim_{x\to 1}\frac{\ln x}{\sqrt{x}-1}$$

$$Ans: \lim_{x\to 1}\frac{\ln x}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\ln x}{(\sqrt{x}-1)(\sqrt{x}+1)}\times(\sqrt{x}+1)=\lim_{x\to 1}\frac{\ln x}{(x-1)}\times\lim_{x\to 1}(\sqrt{x}+1)\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to 1\Rightarrow y\to 0\end{cases}$$

$$=\lim_{y\to 0}\frac{\ln(1+y)}{y}\times(1+1)=1\times 2=2$$

$$Ex: \lim_{x\to 2}\frac{\ln(x-1)}{x-2},\ Ex: \lim_{x\to 0}\frac{\ln(1+x)}{\sqrt{x+1}-1},\ Ex: \lim_{x\to 2}\frac{\ln(x-1)}{(x-2)(x-3)}$$

$$Ex: \lim_{x\to 0}\frac{(x+9)^{\frac{3}{2}}-27}{x}$$

$$Ex: \lim_{x\to 1}\frac{3^{x-1}-1}{x-1}$$

$$Ans: \lim_{x\to 1}\frac{3^{x-1}-1}{x-1}\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to 1\Rightarrow y\to 0\end{cases}$$

$$=\lim_{y\to 0}\frac{3^y-1}{y}=\ln 3$$

$$Ex:\lim_{x\to1}\frac{3^x-3}{x-1}$$

$$Ans:\lim_{x\to1}\frac{3^x-3}{x-1}\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to1\Rightarrow y\to0\end{cases}$$

$$=\lim_{y\to0}\frac{3^{y+1}-3}{y}=\lim_{y\to0}\frac{3\times3^y-3}{y}=3\lim_{y\to0}\frac{3^y-1}{y}=3\ln3=\ln3^3=\ln27$$

$$Ex:\lim_{x\to1}\frac{3^{x-1}-1}{\sqrt{x}-1}$$

$$Ans:\lim_{x\to1}\frac{3^{x-1}-1}{\sqrt{x}-1}=\lim_{x\to1}\frac{3^{x-1}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}\times(\sqrt{x}+1)=\lim_{x\to1}\frac{3^{x-1}-1}{(x-1)}\times\lim_{x\to1}(\sqrt{x}+1)\begin{cases}Let\ x-1=y\Rightarrow x=1+y\\ As\ x\to1\Rightarrow y\to0\end{cases}$$

$$=\lim_{y\to0}\frac{3^y-1}{y}\times2=2\ln3=\ln3^2=\ln9$$

$$Ex:\lim_{x\to1}\frac{2^{x-1}-1}{x-1},Ex:\lim_{x\to1}\frac{2^{x-1}-1}{\sqrt{x}-1}$$

$$Ex:\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

$$Ans:\lim_{x\to\pi}\frac{\sin x}{\pi-x}\begin{cases}Let\ x-\pi=y\Rightarrow x=\pi+y\\ As\ x\to\pi\Rightarrow y\to0\end{cases}$$

$$=\lim_{y\to0}\frac{\sin(\pi+y)}{-y}=\lim_{y\to0}\frac{-\sin y}{-y}=\lim_{y\to0}\frac{\sin y}{y}=1$$

$$Ex:\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$$

$$Ans:\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}\begin{cases}Let\ x-\frac{\pi}{2}=y\Rightarrow x=\frac{\pi}{2}+y\\ As\ x\to\frac{\pi}{2}\Rightarrow y\to0\end{cases}$$

$$=\lim_{y\to0}\frac{1-\sin\left(\frac{\pi}{2}+y\right)}{(-y)^2}=\lim_{y\to0}\frac{1-\cos y}{y^2}=\lim_{y\to0}\frac{2\sin^2\frac{y}{2}}{y^2}=2\lim_{y\to0}\left(\frac{\sin\frac{y}{2}}{y}\right)^2$$

$$=2\lim_{y\to0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\times\frac{1}{2}\right)^2=2\lim_{y\to0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\times\frac{1}{4}=2\times1^2\times\frac{1}{4}=\frac{1}{2}$$

$$Ex:\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\tan x$$

$$Ex:\lim_{x\to1}\frac{x-1}{\ln x}$$

$$Ex:\lim_{x\to0}\frac{2\sin x-\sin 2x}{x^3}$$

$$Ans:\lim_{x\to0}\frac{2\sin x-\sin 2x}{x^3}=\lim_{x\to0}\frac{2\sin x-2\sin x\cos x}{x^3}=\lim_{x\to0}\frac{2\sin x(1-\cos x)}{x^3}$$

$$=2\lim_{x\to0}\frac{\sin x}{x}\times\lim_{x\to0}\frac{(1-\cos x)}{x^2}=2\times1\times\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}=2\times2\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{x}\right)^2=4\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\times\frac{1}{2}\right)^2$$

$$=4\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}=4\times1\times\frac{1}{4}=1$$

$$Ex:\lim_{x\to0}\frac{\tan x-\sin x}{x^3}$$

$$Ans:\lim_{x\to0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}=\lim_{x\to0}\frac{\frac{\sin x-\sin x\cos x}{\cos x}}{x^3}=\lim_{x\to0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\times\lim_{x\to0}\frac{1-\cos x}{x^2}\times\lim_{x\to0}\frac{1}{\cos x}=1\times\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}\times1=2\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{x}\right)^2=2\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\times\frac{1}{2}\right)^2$$

$$=2\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}=2\times1\times\frac{1}{4}=\frac{1}{2}$$

$$Ex:\lim_{x\to0}\frac{1-\cos^3 x}{x\sin 2x}\ ,Ex:\lim_{x\to0}\frac{(1-\cos x)^2}{\tan^3 x-\sin^3 x}\ ,Ex:\lim_{x\to0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$$

$$Ex:Find\ the\ value\ of\ a\ If\ \lim_{x\to\alpha}\frac{\tan a(x-\alpha)}{x-\alpha}=\frac{2}{3}$$

$$Ans: \lim_{x\to\alpha}\frac{\tan a(x-\alpha)}{x-\alpha}=\frac{2}{3}\begin{cases}Let\, x-\alpha=y\Rightarrow x=\alpha+y\\ As\, x\to\alpha\Rightarrow y\to 0\end{cases}$$

$$\Rightarrow \lim_{y\to 0}\frac{\tan ay}{y}=\frac{2}{3}$$

$$\Rightarrow \lim_{y\to 0}\frac{\tan ay}{ay}\times a=\frac{2}{3}\Rightarrow 1\times a=\frac{2}{3}\Rightarrow a=\frac{2}{3}$$

$$Ex: Find\ the\ value\ of\ a\ If\ \lim_{x\to 1}\frac{5^x-5}{(x-1)\ln a}=5$$

$$Ans: \lim_{x\to 1}\frac{5^x-5}{(x-1)\ln a}=5\begin{cases}Let\, x-1=y\Rightarrow x=1+y\\ As\, x\to 1\Rightarrow y\to 0\end{cases}$$

$$\Rightarrow \lim_{y\to o}\frac{5^{1+y}-5}{y\ln a}=5\Rightarrow \lim_{y\to o}\frac{5\times 5^y-5}{y\ln a}=5\Rightarrow \frac{5}{\ln a}\lim_{y\to o}\frac{5^y-1}{y}=5\Rightarrow \frac{1}{\ln a}\ln 5=1$$

$$\Rightarrow \ln 5=\ln a\Rightarrow a=5$$

$$Ex: Ex-46: Find\ the\ value\ of\ a\ If\ \lim_{x\to 2}\frac{\ln(2x-1)}{a(x-2)}=2$$

$$Ans: \lim_{x\to 2}\frac{\ln(2x-1)}{a(x-2)}=2\begin{cases}Let\, x-2=y\Rightarrow x=2+y\\ As\, x\to 2\Rightarrow y\to 0\end{cases}$$

$$\Rightarrow \lim_{y\to 0}\frac{\ln\{2(1+y)-1\}}{ay}=2\Rightarrow \lim_{y\to 0}\frac{\ln\{2+2y-1\}}{ay}=2\Rightarrow \frac{1}{a}\lim_{y\to 0}\frac{\ln(1+2y)}{y}=2$$

$$\Rightarrow \frac{1}{a}\lim_{y\to 0}\frac{\ln(1+2y)}{2y}\times 2=2\Rightarrow \frac{1}{a}\times 1\times 2=2\Rightarrow \frac{2}{a}=2\Rightarrow a=1$$

*Find the value of a*

$$Ex: \lim_{x\to o}\frac{\sin 3x}{\tan ax}=\frac{1}{3},\ Ex: \lim_{x\to o}\frac{e^{ax}-e^{-ax}}{x}=3,$$

## Limits at infinity or infinite limit

**Meaning of $x\to\infty$:** The symbol $x\to\infty$ will be used to mean that x takes very large values.

$$Ex: Show\ that\ \lim_{x\to\infty}\frac{1}{x}=0$$

*Ans:* As x takes very large value , $\frac{1}{x}$ becomes very small value ,(i.e close to zero)

Thus ,when $x\to\infty$,then $\frac{1}{x}\to 0$

Hence $\lim_{x\to\infty}\frac{1}{x}=0$

Note:

$1.\lim_{x\to\infty}\frac{3}{x}=0$

$2.\lim_{x\to\infty}\frac{3}{x^2}=0 \text{ and so on} ----$

**Working rule for finding** $\lim_{x\to\infty} f(x)$ Replace x by $\frac{1}{y}$ in the given function and take the limit as $y\to 0$ OR

In case of a rational function divide the numerator($N^r$ )and the denominator($D^r$ )by the highest power of x .

$$Ex:\lim_{x\to\infty}\frac{5x^2+3x-6}{2x^2-5x+1}$$

$$Ans:\lim_{x\to\infty}\frac{5x^2+3x-6}{2x^2-5x+1}=\lim_{x\to\infty}\frac{5\frac{x^2}{x^2}+3\frac{x}{x^2}-\frac{6}{x^2}}{2\frac{x^2}{x^2}-5\frac{x}{x^2}+\frac{1}{x^2}}\left(Divide\,the N^r \text{ and } D^r by\, x^2\right)$$

$$=\lim_{x\to\infty}\frac{5+\frac{3}{x}-\frac{6}{x^2}}{2-\frac{5}{x}+\frac{1}{x^2}}=\frac{5+0-0}{2-0+0}\left(put\, x=\infty\right)$$

$$=\frac{5}{2}$$

$$Ex:\lim_{x\to\infty}\frac{2x-3}{4x+5}$$

$$Ans:\lim_{x\to\infty}\frac{2x-3}{4x+5}=\lim_{x\to\infty}\frac{2\frac{x}{x}-\frac{3}{x}}{4\frac{x}{x}+\frac{5}{x}}\left(Divide\, x\, in\, both\, N^r \,\&\, D^r\right)$$

$$=\lim_{x\to\infty}\frac{2-\frac{3}{x}}{4+\frac{5}{x}}=\frac{2-0}{4+0}=\frac{2}{4}=\frac{1}{2}$$

$Ex:\lim_{n\to\infty}\frac{2n-3}{4n+5}, Ex:\lim_{x\to\infty}\frac{3x^3-4x^2+2}{4x^3+5x-1}, Ex:\lim_{x\to\infty}\frac{3n^2-4n+1}{4n^2+5n-2}, Ex:\lim_{n\to\infty}\frac{2n}{4n+5}$

$Ex:\lim_{x\to\infty}\frac{3x^3-4x^2+2}{4x^2+5x-3}$

$$Ans:\lim_{x\to\infty}\frac{3x^3-4x^2+2}{4x^2+5x-3}=\lim_{x\to\infty}\frac{3\frac{x^3}{x^3}-4\frac{x^2}{x^3}+\frac{2}{x^3}}{4\frac{x^2}{x^3}+5\frac{x}{x^3}-\frac{3}{x^3}}\left(Divide\, x^3\, in\, both\, N^r\, \&\, D^r\right)$$

$$=\lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{2}{x^3}}{\frac{4}{x}+\frac{5}{x^2}-\frac{3}{x^3}}=\frac{3-0+0}{0+0-0}=\frac{3}{0}=\infty$$

$Ex:\lim_{x\to\infty}\frac{3x^2-4x+2}{4x^3+5x^2-3}$

$$Ans:\lim_{x\to\infty}\frac{3x^2-4x+2}{4x^3+5x^2-3}=\lim_{x\to\infty}\frac{3\frac{x^2}{x^3}-4\frac{x}{x^3}+\frac{2}{x^3}}{4\frac{x^3}{x^3}+5\frac{x^2}{x^3}-\frac{3}{x^3}}\left(Divide\, x^3\, in\, both\, N^r\, \&\, D^r\right)$$

$$=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{4}{x^2}+\frac{2}{x^3}}{4+\frac{5}{x}-\frac{3}{x^3}}=\frac{0-0+0}{4+0-0}=0$$

$Ex:\lim_{n\to\infty}\frac{1+2+3+\cdots\cdots+n}{n^2}$

$$Ans:\lim_{n\to\infty}\frac{1+2+3+\cdots\cdots+n}{n^2}=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\lim_{n\to\infty}\frac{n+1}{2n}\left(Divide\, n\, in\, both\, N^r\, \&\, D^r\right)$$

$$=\lim_{n\to\infty}\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}=\frac{1+0}{2}=\frac{1}{2}$$

$Ex:\lim_{n\to\infty}\frac{1^3+2^3+3^3+\cdots\cdots+n^3}{n^4}$

$$Ans:\lim_{n\to\infty}\frac{1^3+2^3+3^3+\cdots\cdots+n^3}{n^4}=\lim_{n\to\infty}\frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4}=\lim_{n\to\infty}\frac{\{n(n+1)\}^2}{4n^4}=\lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+2\frac{n}{n^2}+\frac{1}{n^2}}{4\frac{n^2}{n^2}} \left(Divide\, n^2\, in\, both\, N^r\ \&\ D^r\right)$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4} = \frac{1+0+0}{4} = \frac{1}{4}$$

$$Ex: \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}$$

$$Ans: \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}$$

$$= \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 2\lim_{n\to\infty}\left\{1-\left(\frac{1}{2}\right)^{n+1}\right\} = 2(1-0)\left(\lim_{n\to\infty} x^n = 0\, if\, -1<x<1\right)$$

$$= 2$$

$$Ex: \lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\cdots+\frac{1}{3^n}}$$

$$Ans: \lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\cdots+\frac{1}{3^n}} = \frac{\lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}}{\lim_{n\to\infty} 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\cdots+\frac{1}{3^n}}$$

$$= \frac{\lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}}{\lim_{n\to\infty} \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}} = \frac{\lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}}{\lim_{n\to\infty} \frac{1-\left(\frac{1}{3}\right)^{n+1}}{\frac{2}{3}}} = \frac{2\lim_{n\to\infty}\left\{1-\left(\frac{1}{2}\right)^{n+1}\right\}}{\frac{3}{2}\lim_{n\to\infty}\left\{1-\left(\frac{1}{3}\right)^{n+1}\right\}} = \frac{4}{3}\frac{(1-0)}{(1-0)} = \frac{4}{3}$$

$$Ex: \lim_{n\to\infty} 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\cdots+\frac{1}{3^n}$$

## Test the existence of the limit $\lim_{x \to a} f(x)$

The limit of a function f(x) is said to be exist at x=a ( i.e $\lim_{x \to a} f(x)\, exist$ )

If Left hand limit (LHL) = Right hand limit (RHL)

Left hand limit (LHL)

$$\lim_{h \to 0} f(a-h)\ where\ h\ is\ small\ (+ve)\ quantity$$
$$or\ \lim_{x \to a^-} f(x)$$

Right hand limit (RHL)

$$\lim_{h \to 0} f(a+h)\ where\ h\ is\ small\ (+ve)\ quantity$$
$$or\ \lim_{x \to a^+} f(x)$$

Note:If L.H.L $\neq$ R.H.L then the limit of a function does not exist

Ex: Test the existence of the limit $\lim_{x \to 1} |x-1|$

$Ans: \lim_{x \to 1} |x-1|$

$Here\ a = 1,\ f(x) = |x-1|$

$L.H.L \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(\underline{1-h})$ (put $x = 1-h$ in the given function $f(x) = |x-1|$)

$= \lim_{h \to 0} |1-h-1| = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0$

$R.H.L \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(\underline{1+h})$ (put $x = 1+h$ in the given function $f(x) = |x-1|$)

$= \lim_{h \to 0} |1+h-1| = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$

$So\ L.H.L = R.H.L$

Hence the limit of the function exist.

Ex: Test the existence of the limit $\lim_{x \to 0} |x|$

$Ans: \lim_{x \to 0} |x|$

$Here\ a = 0,\ f(x) = |x|$

$L.H.L \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(\underline{0-h})$ (put $x = 0-h$ in the given function $f(x) = |x|$)

$= \lim_{h \to 0} |0-h| = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0$

$R.H.L \lim_{h\to0} f(a+h) = \lim_{h\to0} f(\underline{0+h})$ (put $x = 0+h$ in the given function $f(x) = |x|$)

$= \lim_{h\to0} |0+h| = \lim_{h\to0} |h| = \lim_{h\to0} h = 0$

So $L.H.L = R.H.L$

Hence the limit of the function exist.

Ex: Test the existence of the limit $\lim_{x\to0} \frac{|x|}{x}$

*Ans* : $\lim_{x\to0} \frac{|x|}{x}$

Here $a = 0,\ f(x) = \frac{|x|}{x}$

$L.H.L \lim_{h\to0} f(a-h) = \lim_{h\to0} f(\underline{0-h})$ $\left(\text{put } x = 0-h \text{ in the given function } f(x) = \frac{|x|}{x}\right)$

$= \lim_{h\to0} \frac{|0-h|}{0-h} = \lim_{h\to0} \frac{|-h|}{-h} = \lim_{h\to0} \frac{h}{-h} = \lim_{h\to0} -1 = -1$

$R.H.L \lim_{h\to0} f(a+h) = \lim_{h\to0} f(\underline{0+h})$ $\left(\text{put } x = 0+h \text{ in the given function } f(x) = \frac{|x|}{x}\right)$

$= \lim_{h\to0} \frac{|0+h|}{0+h} = \lim_{h\to0} \frac{|h|}{h} = \lim_{h\to0} \frac{h}{h} = \lim_{h\to0} 1 = 1$

So $L.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex:Test the existence of the limit

(a) $\lim_{x\to2} |x-2|$ (b) $\lim_{x\to-1} |x+1|$ (c) $\lim_{x\to0} \frac{x}{|x|}$ (d) $\lim_{x\to1} \frac{|x-1|}{x-1}$ (e) $\lim_{x\to1} \frac{|x-2|}{x-2}$

Ex:Test the existence of the limit $\lim_{x\to2} [x]$

*Ans* : $\lim_{x\to2} [x]$

Here $a = 2,\ f(x) = [x]$

$L.H.L \lim_{h\to0} f(a-h) = \lim_{h\to0} f(2-h) = \lim_{h\to0} [2-h] = \lim_{h\to0} 1 = 1$

$R.H.L \lim_{h\to0} f(a+h) = \lim_{h\to0} f(2+h) = \lim_{h\to0} [2+h] = \lim_{h\to0} 2 = 2$

So $L.H.L \neq R.H.L$

Hence the limit of the function does not exist.

Ex:Test the existence of the limit $\lim_{x\to 0}[x]$

$Ans:\lim_{x\to 0}[x]$

Here $a=0,\ f(x)=[x]$

$L.H.L\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}[0-h]=\lim_{h\to 0}-1=-1$

$R.H.L\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}[0+h]=\lim_{h\to 0}0=0$

$So L.H.L\neq R.H.L$

Hence the limit ofthe function does not exist.

Ex:Test the existence of the limit $\lim_{x\to 1}[x+1]$

$Ans:\lim_{x\to 1}[x+1]$

Here $a=1,\ f(x)=[x+1]$

$L.H.L\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}[1-h+1]=\lim_{h\to 0}[2-h]=1$

$R.H.L\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}[1+h+1]=\lim_{h\to 0}[2+h]=2$

$So L.H.L\neq R.H.L$

Hence the limit ofthe function does not exist.

Ex:Test the existence of the limit $\lim_{x\to\sqrt{2}}[x]$

$Ans:\lim_{x\to\sqrt{2}}[x]$

Here $a=\sqrt{2},\ f(x)=[x]$

$L.H.L\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f\left(\sqrt{2}-h\right)=\lim_{h\to 0}\left[\sqrt{2}-h\right]=\lim_{h\to 0}1=1$

$R.H.L\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f\left(\sqrt{2}+h\right)=\lim_{h\to 0}\left[\sqrt{2}+h\right]=\lim_{h\to 0}1=1$

$So L.H.L=R.H.L$

Hence the limit of the function exist.

Ex:Test the existence of the limit

$(a)\lim_{x\to 1}[x]\ (b)\lim_{x\to -2}[x]\ (c)\lim_{x\to n}[x]$ if n be an integer $(d)\lim_{x\to\sqrt{3}}[x]$

Ex:Test the existence of the limit $\lim_{x\to 1} f(x)$

$$If\ \ f(x)=\begin{cases}2x+1, x<1\\ x-3, x\geq 1\end{cases}$$

$Ans:$

$L.H.L\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(1-h)$ $\quad\{Here\ f(x)=2x+1 and\ put\ x=1-h\}$

$=\lim_{h\to 0}\{2(1-h)+1\}=\lim_{h\to 0}\{2-2h+1\}=\lim_{h\to 0}3-2h=3$

$R.H.L \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(1+h)$ $\{Here\ f(x) = x-3\, and\ put\ x = 1+h\}$

$= \lim_{h \to 0}\{(1+h)-3\} = \lim_{h \to 0}(h-2) = -2$

$So\ L.H.L \neq R.H.L$

Hence the limit does not exist.

Ex:Test the existence of the limit $\lim_{x \to 2} f(x)$

$$If\ \ f(x) = \begin{cases} 2x-3, x \leq 2 \\ x^2-3, x > 2 \end{cases}$$

Ex:Test the existence of the limit $\lim_{x \to 2} f(x)$

$$If\ \ f(x) = \begin{cases} x, x > 0 \\ -x, x < 0 \\ 0, x = 0 \end{cases}$$

## CONTINUITY

**A function f(x) is said to be continuous at a point x=a if**

$$\left\{ \begin{array}{l} \lim_{x \to a} f(x) \quad = \quad f(a) \\ \text{limiting value} = \text{functional value} \end{array} \right\}$$

$OR$

$$\left\{ \begin{array}{l} L.H.L = R.H.L = f(a) \\ \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) = f(a) \end{array} \right\}$$

## DISCONTINUITY

**A function f(x) is said to be discontinuous at a point x=a if**

1. $f(a)$ is not defined
2. $\lim_{x \to a} f(x) \neq f(a)$
3. $L.H.L = R.H.L \neq f(a)$
4. $L.H.L \neq R.H.L = f(a)$
5. $L.H.L = f(a) \neq R.H.L$
6. $L.H.L \neq R.H.L \neq f(a)$
7. $\lim_{x \to a} f(x)$ does not exist

Ex:Test the continuity of the function

$$f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & x\neq 0\\ 3 & ,x=0\end{cases} \qquad at\ x=0.$$

$Ans: At\ x=0\ ,\ f(x)=3 \Rightarrow f(0)=3$

$at\ x\neq 0,\ f(x)=\dfrac{\sin 3x}{x}$

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\times 3=1\times 3=3$

$So \lim\limits_{x\to 0} f(x)=f(0)=3$

Hence the function is continuous at $x=0$.

Ex:Find the value of a if the function $f(x)=\begin{cases}\dfrac{\sin ax}{2x}, & x\neq 0\\ 3 & ,x=0\end{cases}$ is continuous at $x=0$.

$Ans: At\ x=0\ ,\ f(x)=3 \Rightarrow f(0)=3$

$at\ x\neq 0,\ f(x)=\dfrac{\sin ax}{2x}$

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\sin ax}{2x}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin ax}{x}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}\times a=\dfrac{1}{2}\times 1\times a=\dfrac{a}{2}$

Since the function is continuous at $x=0$.

$So \lim\limits_{x\to 0} f(x)=f(0)$

$\Rightarrow \dfrac{a}{2}=3 \Rightarrow a=6$

Ex:Test the continuity of the function

$$f(x)=\begin{cases}\dfrac{x^7-1}{x-1}, & x\neq 1\\ 7 & ,x=1\end{cases} \qquad at\ x=1.$$

$Ans: Ans: At\ x=1,\ f(x)=7 \Rightarrow f(1)=7$

$at\ x\neq 1,\ f(x)=\dfrac{x^7-1}{x-1}$

$\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\dfrac{x^7-1}{x-1}=\lim\limits_{x\to 1}\dfrac{x^7-1^7}{x-1}=7\times 1^6=7$

$So \lim\limits_{x\to 1} f(x)=f(1)=7$

Hence the function is continuous at $x=1$.

Ex:Test the continuity of the function

$$f(x)=\begin{cases}(1+3x)^{\frac{1}{x}}, & x\neq 0\\ e^3 & ,x=0\end{cases} \qquad at\ x=0.$$

$Ans: Ans: At\ x=0,\ f(x)=e^3 \Rightarrow f(0)=e^3$

$at\ x\neq 0,\ f(x)=(1+3x)^{\frac{1}{x}}$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(1+3x)^{\frac{1}{x}}=\lim_{x\to 0}(1+3x)^{\frac{1}{3x}\times 3}=\left\{\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right\}^3=e^3$$

$So \lim_{x\to 0} f(x)=f(0)=e^3$

Hence the function is continuous at $x=0$.

Ex:Test the continuity of the function

$$(a)\ f(x)=\begin{cases}\dfrac{\sin 2x}{\tan 3x}, & x\neq 0\\ \dfrac{2}{3} & ,x=0\end{cases} \qquad at\ x=0$$

$$(b)\ f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a}, & x\neq a\\ 2a & ,x=a\end{cases} \qquad at\ x=a$$

$$(c)\ f(x)=\begin{cases}\dfrac{e^{5x}-e^{-2x}}{x}, & x\neq 0\\ 5 & ,x=0\end{cases} \qquad at\ x=0$$

$$(d)\ f(x)=\begin{cases}\dfrac{a^x-b^x}{x}, & x\neq 0\\ \ln\dfrac{a}{b} & ,x=0\end{cases} \qquad at\ x=0$$

$$(e)\ f(x)=\begin{cases}\dfrac{\ln(1+3x)}{x}, & x\neq 0\\ 3 & ,x=0\end{cases} \qquad at\ x=0$$

$$(f)\ f(x)=\begin{cases}(1+2x)^{\frac{1}{x}}, & x\neq 0\\ e^2 & ,x=0\end{cases} \qquad at\ x=0$$

Ex:Test the continuity of the function

$$f(x)=\begin{cases}\frac{|x|}{x}, x\neq 0 \\ 0 \quad , x=0\end{cases} \quad at\ x=0$$

Ans: at $x=0$, $f(x)=0 \Rightarrow f(0)=0$

$at\ x\neq 0,\ f(x)=\frac{|x|}{x}$

$L.H.L\ \lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{|0-h|}{0-h}=\lim_{h\to 0}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$

$R.H.L\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{|0+h|}{0+h}=\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=1$

$So\ L.H.L\neq R.H.L\neq f(0)$

Hence the function is discontinuous.

Ex:Show that the function $\mathrm{f}(\mathrm{x})=[x]$ is not continuous at $x=2$.

$Ans: At\ x=2, \mathrm{f}(\mathrm{x})=[x]$

$\Rightarrow f(2)=[2]=2$

$L.H.L\ \lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0}[2-h]=\lim_{h\to 0}1=1$

$R.H.L\ \lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}[2+h]=\lim_{h\to 0}2=2$

$So\ L.H.L\neq R.H.L=f(0)$

Hence the function is discontinuous.

Ex:Test the continuity of the function

$$\mathrm{f}(\mathrm{x})=\begin{cases}2x+3, x<1 \\ x-4 \quad, x\geq 1\end{cases} \quad at\ x=1$$

$Ans: at\ x=1,\ f(x)=x-4 \Rightarrow f(1)=1-4=-3$

$L.H.L\ \lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(1-h) \quad , f(x)=2x+3,$ (put $x=1-h$ in the function $f(x)=2x+3$)

$=\lim_{h\to 0}\{2(1-h)+3\}=\lim_{h\to 0}2-2h+3=5$

$=\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(1+h) \quad , f(x)=x-4,$ (put $x=1+h$ in the function $f(x)=x-4$)

$=\lim_{h\to 0}\{(1+h)-4\}=\lim_{h\to 0}1-4=-3$

$L.H.L\neq R.H.L=f(1)$

Hence the function is discontinuous.

Ex: Test the continuity of the function

$$f(x)=\begin{cases}2x-1, & x\le 1\\ x+2, & x>1\end{cases}\quad at\ x=1$$

Ex: Test the continuity of the function

$$f(x)=\begin{cases}2x-1, & x<0\\ 3, & x=0\\ x+2, & x>0\end{cases}\quad at\ x=0$$

Ex: Test the continuity of the function

$$f(x)=\begin{cases}x, & x>0\\ 0, & x=0\\ -x, & x<0\end{cases}\quad at\ x=0$$

Ex: Find the value of a if the function

$$f(x)=\begin{cases}\dfrac{\sin ax}{\tan x}, & x\ne 0\\ \dfrac{1}{a}, & x=0\end{cases}\quad \text{is continuous at } x=0$$

Ex: Find the value of a and b if the function

$$f(x)=\begin{cases}ax^2+b, & x<1\\ 1, & x=1\\ 2ax-b, & x>1\end{cases}\quad \text{is continuous at } x=1$$

(hints-see the semester question below)

Formula: $\lim_{x\to 0}\dfrac{(1+x)^n-1}{x}=n$

proof: $\lim_{x\to 0}\dfrac{(1+x)^n-1}{x}$ $\quad let\ 1+x=y\Rightarrow x=y-1$

$As\ x\to 0\Rightarrow y\to 1$

$$=\lim_{y\to 1}\frac{y^n-1^n}{y-1}=n\times 1^{n-1}=n$$

Note: $\lim_{x\to 0}\dfrac{(1-x)^n-1}{x}=-n$

Ex: $\lim_{x\to 0}\dfrac{(1+x)^{\frac{1}{2}}-1}{x}=\dfrac{1}{2}$ (by using formula)

$$\text{Ex}: \lim_{x\to 0}\frac{(1-x)^{\frac{1}{2}}-1}{x} = -\frac{1}{2}(\text{by using formula})$$

$$\text{Ex}: \lim_{x\to 0}\frac{(1+x)^{\frac{3}{2}}-1}{x} = \frac{3}{2}(\text{by using formula})$$

$$\text{Ex}: \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

$$Ans: \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{\sqrt{1+x}-1+1-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)-\left(\sqrt{1-x}-1\right)}{x}$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)}{x} - \lim_{x\to 0}\frac{\left(\sqrt{1-x}-1\right)}{x} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1}{x} - \lim_{x\to 0}\frac{(1-x)^{\frac{1}{2}}-1}{x}$$

$$= \frac{1}{2}-\left(-\frac{1}{2}\right) = \frac{1}{2}+\frac{1}{2} = 1$$

$$\text{Ex}: \lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$$

$$Ans: \lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x} = \lim_{x\to 0}\frac{\sqrt[3]{1+x}-1+1-\sqrt[3]{1-x}}{x} = \lim_{x\to 0}\frac{\left(\sqrt[3]{1+x}-1\right)-\left(\sqrt[3]{1-x}-1\right)}{x}$$

$$= \lim_{x\to 0}\frac{\left(\sqrt[3]{1+x}-1\right)}{x} - \lim_{x\to 0}\frac{\left(\sqrt[3]{1-x}-1\right)}{x} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-1}{x} - \lim_{x\to 0}\frac{(1-x)^{\frac{1}{3}}-1}{x}$$

$$= \frac{1}{3}-\left(-\frac{1}{3}\right) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

$$\text{Ex}: \frac{\lim_{x\to 0}\sqrt{1+x}-\sqrt{1-x}}{\lim_{x\to 0}\sqrt[3]{1+x}-\sqrt[3]{1-x}}$$

$$Ans: \text{Ex}: \frac{\lim_{x\to 0}\sqrt{1+x}-\sqrt{1-x}}{\lim_{x\to 0}\sqrt[3]{1+x}-\sqrt[3]{1-x}} = \frac{\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}}{\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}} = \frac{\lim_{x\to 0}\frac{\sqrt{1+x}-1+1-\sqrt{1-x}}{x}}{\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1+1-\sqrt[3]{1-x}}{x}}$$

$$= \frac{\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)-\left(\sqrt{1-x}-1\right)}{x}}{\lim_{x\to 0}\frac{\left(\sqrt[3]{1+x}-1\right)-\left(\sqrt[3]{1-x}-1\right)}{x}} = \frac{\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)}{x}-\lim_{x\to 0}\frac{\left(\sqrt{1-x}-1\right)}{x}}{\lim_{x\to 0}\frac{\left(\sqrt[3]{1+x}-1\right)}{x}-\lim_{x\to 0}\frac{\left(\sqrt[3]{1-x}-1\right)}{x}}$$

$$=\frac{\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1}{x}-\lim_{x\to 0}\frac{(1-x)^{\frac{1}{2}}-1}{x}}{\lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-1}{x}-\lim_{x\to 0}\frac{(1-x)^{\frac{1}{3}}-1}{x}}=\frac{\frac{1}{2}-\left(-\frac{1}{2}\right)}{\frac{1}{3}-\left(-\frac{1}{3}\right)}=\frac{\frac{1}{2}+\frac{1}{2}}{\frac{1}{3}+\frac{1}{3}}=\frac{1}{\left(\frac{2}{3}\right)}=\frac{3}{2}$$

## SEMESTER QUESTIONS

**Q.1** Evaluate $\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos^2 x}{1-\sin x}\right)$

$$Ans:\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos^2 x}{1-\sin x}\right)=\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin^2 x}{1-\sin x}\right)=\lim_{x\to\frac{\pi}{2}}\left(\frac{(1-\sin x)(1+\sin x)}{1-\sin x}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}(1+\sin x)=1+\sin\frac{\pi}{2}=1+1=2$$

**Q.2** Evaluate $\lim_{x\to o}\left(\frac{\cos ecx-\cot x}{x}\right)$

$$Ans:\lim_{x\to o}\left(\frac{\cos ecx-\cot x}{x}\right)=\lim_{x\to o}\left(\frac{\frac{1}{\sin x}-\frac{\cos x}{\sin x}}{x}\right)=\lim_{x\to o}\left(\frac{1-\cos x}{x\sin x}\right)=\lim_{x\to o}\left(\frac{2\sin^2\frac{x}{2}}{x2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$=\lim_{x\to o}\left(\frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}}\right)=\lim_{x\to o}\left(\frac{\sin\frac{x}{2}}{x}\right)\times\lim_{x\to o}\left(\frac{1}{\cos\frac{x}{2}}\right)=\lim_{x\to o}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\times\frac{1}{2}\times 1=1\times\frac{1}{2}=\frac{1}{2}$$

**Q.3 Discuss the continuity of the function**

$$f(x)=\begin{cases}3x-2\ when x\le 0\\ x+1\quad when\, x>0\end{cases}\ at\ x=0$$

Ans:

$$At\ x=0,\ f(x)=3x-2\ \Rightarrow\ f(0)=3\times 0-2=-2$$

$L.H.L$

$$\lim_{h\to 0}f(a-h)=\lim_{h\to 0}f(0-h),\ f(x)=3x-2$$

$= \lim_{h\to 0}\{3(0-h)-2\} = -2$ ,

$R.H.L$

$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}(0+h+1) = 1$ , $f(x) = x+1$

So $L.H.L = f(0) \neq R.H.L$

Hence the function is discontinuous.

**Q.4 Evaluate** $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)$

$$Ans: \lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right) = \lim_{x\to\infty} \frac{\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}\left(x+1-x\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+\frac{1}{x}}+1\right)} = \lim_{x\to\infty} \frac{1}{\left(\sqrt{1+\frac{1}{x}}+1\right)} = \frac{1}{2}$$

**Q.5 Evaluate** $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

Ans: $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$ let $\sin^{-1} x$ =y $\Rightarrow$ x=siny

As x $\to$ 0 $\Rightarrow$ y $\to$ 0

$$= \lim_{y\to 0} \frac{y}{\sin y} = 1$$

**Q.6 Evaluate** $\lim_{x\to 1} \frac{\log_e(2x-1)}{x-1}$

Ans:

$\lim_{x\to 1} \frac{\log_e(2x-1)}{x-1}$ Let $x-1 = y \Rightarrow x = 1+y$

As $x \to 1 \Rightarrow y \to 0$

$$=\lim_{y\to 0}\frac{\log_e(2(1+y)-1)}{y}=\lim_{y\to 0}\frac{\log_e(2+2y-1)}{y}=\lim_{y\to 0}\frac{\log_e(1+2y)}{y}$$

$$=\lim_{y\to 0}\frac{\log_e(1+2y)}{2y}\times 2=1\times 2=2$$

**Q.7 If** $f(x)=\begin{cases} ax^2+b, x<1 \\ 1 \quad\quad\;\, ,x=1 \\ 2ax-b, x>1 \end{cases}$ is continuous at x=1 ,then find a and b.

Ans: at x=1 , f(x)=1 $\Rightarrow$ f(1)=1

L.H.L

$$\begin{aligned}&\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(1-h)\\ &=\lim_{h\to 0} a(1-h)^2+b=a+b\end{aligned} \quad , \; f(x)=ax^2+b$$

R.H.L

$$\begin{aligned}&\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(1+h)\\ &=\lim_{h\to 0} 2a(1+h)-b=2a-b\end{aligned} \quad , \; f(x)=2ax-b$$

Since thefunction is continuous, so L.H.L= R.H.L= f(1)

So

$$\begin{aligned}a+b &=1\\ 2a-b&=1\end{aligned}$$

$\Rightarrow 3a=2\Rightarrow a=\frac{2}{3}$ so $a+b=1\Rightarrow b=1-\frac{2}{3}=\frac{1}{3}$

**Q.8 Evaluate** $\lim_{x\to\infty} x\sin\frac{1}{x}$

Ans: $\lim_{x\to\infty} x\sin\frac{1}{x}$ let $x=\frac{1}{y}\Rightarrow y=\frac{1}{x}$

As $x\to\infty\Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{1}{y}\sin y=\lim_{y\to 0}\frac{\sin y}{y}=1$$

**Q.9 Evaluate** $\lim_{x\to 0}\frac{(x+9)^{\frac{3}{2}}-27}{x}$

Ans: $\lim_{x\to 0}\frac{(x+9)^{\frac{3}{2}}-27}{x}$ Let $x+9=y \Rightarrow x=y-9$

As $x\to 0 \Rightarrow y\to 9$

$$=\lim_{y\to 9}\frac{y^{\frac{3}{2}}-9^{\frac{3}{2}}}{y-9}=\frac{3}{2}\times 9^{\frac{3}{2}-1}=\frac{3}{2}\times 9^{\frac{1}{2}}=\frac{3}{2}\times 3=\frac{9}{2}$$

**Q.10 If** $f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 1 \quad , x=0\end{cases}$ examine the continuity of f(x) at x=0.

Ans: at x=0,f(x)=1 $\Rightarrow$ f(0)=1

At $x\neq 0,\ f(x)=\frac{|x|}{x}$

L.H.L $\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{|0-h|}{0-h}=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}-1=-1$

R.H.L $\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{|0+h|}{0+h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$

LHL $\neq$ RHL=f(a)

Hence the function is not continuous at x=0.

**Q.11 Evaluate** $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

Ans: $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\frac{\lim_{x\to 0}\frac{\sin 3x}{x}}{\lim_{x\to 0}\frac{\sin 5x}{x}}=\frac{\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3}{\lim_{x\to 0}\frac{\sin 5x}{5x}\times 5}=\frac{1\times 3}{1\times 5}=\frac{3}{5}$

**Q.12 Examine the continuity of the function f(x) at x=0 defined by** $f(x)=\begin{cases}\frac{\sin 2x}{x}, x\neq 0\\ 2 \quad\quad , x=0\end{cases}$ $at\ x=0$.

Ans:At x=0 , f(x)=2 $\Rightarrow$ f(0)=2

At $x \neq 0,\ f(x) = \dfrac{\sin 2x}{x}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \times 2 = 1\times 2 = 2$$

So $\lim_{x\to 0} f(x)$ = f(0)=2

Hence the function is continuous.

**Q.13 Show that $\lim_{x\to 0} \dfrac{x}{|x|}$ does not exit.**

Ans: Here a=0 , $f(x) = \dfrac{x}{|x|}$

L.H.L $\lim_{h\to o} f(a-h) = \lim_{h\to o} f(0-h) = \lim_{h\to o} \dfrac{0-h}{|0-h|} = \lim_{h\to o} \dfrac{-h}{h} = \lim_{h\to o} -1 = -1$

R.H.L $\lim_{h\to o} f(a+h) = \lim_{h\to o} f(0+h) = \lim_{h\to o} \dfrac{0+h}{|0+h|} = \lim_{h\to o} \dfrac{h}{h} = \lim_{h\to o} 1 = 1$

So LHL $\neq$ RHL , Hence the limit does not exist.

**Q.14 If** $\lim_{x\to 0} \dfrac{\sin 3x}{\tan ax} = 1$ , find the value of a.

Ans:

$$\lim_{x\to 0} \frac{\sin 3x}{\tan ax} = 1 \Rightarrow \frac{\lim_{x\to 0} \frac{\sin 3x}{x}}{\lim_{x\to 0} \frac{\tan ax}{x}} = 1 \Rightarrow \frac{\lim_{x\to 0} \frac{\sin 3x}{3x} \times 3}{\lim_{x\to 0} \frac{\tan ax}{ax} \times a} = 1$$

$$\Rightarrow \frac{1\times 3}{1\times a} = 1 \Rightarrow \frac{3}{a} = 1 \Rightarrow a = 3$$

**Q.15 If** $f(x) = \begin{cases} (1+3x)^{\frac{1}{3}}, & x \neq 0 \\ e^3 & , x = 0 \end{cases}$ then examine the continuity of f(x) at x=0.

Ans: At x=0 , $f(x) = e^3 \Rightarrow f(0) = e^3$

At x=0 , $f(x)=(1+3x)^{\frac{1}{x}}$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(1+3x)^{\frac{1}{x}}=\lim_{x\to 0}(1+3x)^{\frac{1}{3x}\times 3}=\left\{\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right\}^3=e^3$$

So $\lim_{x\to 0} f(x)$=f(0)

Hence the function is continuous.

**Q.16 Evaluate** $\lim_{x\to 0}\frac{x-x\cos 2x}{\sin^3 2x}$

$$Ans:\lim_{x\to 0}\frac{x-x\cos 2x}{\sin^3 2x}=\lim_{x\to 0}\frac{x(1-\cos 2x)}{(\sin 2x)^3}=\lim_{x\to 0}\frac{x2\sin^2 x}{(2\sin x\cos x)^3}=\lim_{x\to 0}\frac{x2\sin^2 x}{8\sin^3 x\cos^3 x}$$

$$=\lim_{x\to 0}\frac{x}{4\sin x\cos^3 x}=\frac{1}{4}\lim_{x\to 0}\frac{x}{\sin x}\times\lim_{x\to 0}\frac{1}{\cos^3 x}=\frac{1}{4}\times 1\times\frac{1}{1}=\frac{1}{4}$$

**Q.17 Evaluate** $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$Ans:\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x}\right)^2=2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\times\frac{1}{2}\right)^2$$

$$=2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}=2\times 1^2\times\frac{1}{4}=\frac{1}{2}$$

**Q.18 Determine the value of** $\alpha$ such that $f(x)=\begin{cases}\frac{x^2-9}{x-3}, & x\neq 3\\ \alpha & ,x=3\end{cases}$ is continuous at x=3.

Ans: At x=3, $f(x)=\alpha \Rightarrow f(3)=\alpha$

**At** $x\neq 0, f(x)=\frac{x^2-9}{x-3}$

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{x^2-3^2}{x-3}=2\times 3^1=6$$

Since the function is continuous at x=3 . So $\lim_{x\to 3} f(x)$ =f(3)

$\Rightarrow 6=\alpha$ .

**Q.19 Evaluate** $\lim_{x\to 1}\frac{\ln(2x-1)}{x-1}$

$Ans: \lim_{x\to 1}\frac{\ln(2x-1)}{x-1}$ Let $x-1=y\Rightarrow x=1+y$

As $x\to 1\Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{\ln(2(1+y)-1)}{y}=\lim_{y\to 0}\frac{\ln(2+2y-1)}{y}=\lim_{y\to 0}\frac{\ln(1+2y)}{y}$$

$$=\lim_{y\to 0}\frac{\ln(1+2y)}{2y}\times 2=1\times 2=2$$

**Q.20 Evaluate** $\lim_{x\to 0}\frac{\sin x^0}{x}$

Ans: $$\lim_{x\to 0}\frac{\sin x^0}{x}=\lim_{x\to 0}\frac{\sin\frac{\pi}{180}x}{x}=\lim_{x\to 0}\frac{\sin\frac{\pi}{180}x}{\frac{\pi}{180}x}\times\frac{\pi}{180}=1\times\frac{\pi}{180}=\frac{\pi}{180}$$

**Q.21 Evaluate** $\lim_{n\to\infty}\frac{1^2+2^2+3^2+.......+n^2}{n^3}$

$$Ans: \lim_{n\to\infty}\frac{1^2+2^2+3^2+.......+n^2}{n^3}=\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$=\lim_{n\to\infty}\frac{2n^2+2n+n+1}{6n^2}=\lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}=\lim_{n\to\infty}\frac{\frac{2n^2}{n^2}+\frac{3n}{n^2}+\frac{1}{n^2}}{\frac{6n^2}{n^2}}$$

$$=\lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}=\frac{2+0+0}{6}=\frac{2}{6}=\frac{1}{3}$$

**Q.22** ***Examine the continuity of the function***

$$f(x)\, at\, x=0\, defined\, by\, f(x)=\begin{cases} 2x+1 & when\, x<1 \\ 0 & when\, x=0 \\ x^2-1 & when\, x>1 \end{cases}$$

$$Ans: At\, x=0,\, f(x)=0 \Rightarrow f(0)=0$$

$$L.H.L \lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(0-h),\, f(x)=2x+1$$

$$=\lim_{h\to 0}\{2(0-h)+1\}=1$$

$$R.H.L\ \lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(0+h),\, f(x)=x^2-1$$

$$=\lim_{h\to 0}\{(0-h)^2-1\}=-1$$

$$So\, L.H.L \neq R.H.L \neq f(0)$$

$$Hence\, the\;\, function\, is\, not\, continuous$$

**Q.23**

$$Evaluate \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

$$Ans: \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0}\frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}=\lim_{x\to 0}\frac{1+x-1+x}{x(\sqrt{1+x}+\sqrt{1-x})}=\lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2}{2}=1$$

**Q.24**

$$Evaluate \lim_{x\to 0}\frac{\tan 5x}{\tan 7x}$$

$$Ans: \lim_{x\to 0}\frac{\tan 5x}{\tan 7x}=\lim_{x\to 0}\frac{\frac{\tan 5x}{x}}{\frac{\tan 7x}{x}}=\lim_{x\to 0}\frac{\frac{\tan 5x}{5x}\times 5}{\frac{\tan 7x}{7x}\times 7}=\frac{1\times 5}{1\times 7}=\frac{5}{7}$$

**Q.25**

$$Test\ the\ continuity\ of\ the\ function\ at\ x = 0\ if$$

$$f(x) = \begin{cases} x\sin\frac{1}{x}\ when\ x \neq 0 \\ 0 \qquad when\ x = 0 \end{cases}$$

$$Ans: at\ x = 0,\ f(x) = 0 \Rightarrow f(0) = 0$$

$$at\ x \neq 0,\ f(x) = x\sin\frac{1}{x}$$

$$We\ have\ -1 \leq \sin\frac{1}{x} \leq 1 \Rightarrow -x \leq x\sin\frac{1}{x} \leq x$$

$$\Rightarrow \lim_{x\to 0} -x \leq \lim_{x\to 0} x\sin\frac{1}{x} \leq \lim_{x\to 0} x$$

$but \lim_{x\to 0} -x = 0\ and\ \lim_{x\to 0} x = 0\ by$ using sandwich theorem we get

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

$$So \lim_{x\to 0} x\sin\frac{1}{x} = f(0)$$

$$Hence\ the\ function\ is\ continuous$$