LAPLACE TRANSFORM

Def^n: Let f(t) be the real valued function of t, t > 0. Then Laplace transform of f(t) is given by $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$, where s is parameter may be real or complex

Formulas: (Laplace Transformation of some simple functions)

(i) Laplace Transform of constant function :

Let f(t)=k ,where k is constant

Then Laplace transform of f(t) is given by

L{f(t)}=L(k)= $\frac{k}{s}$ *

Ex : L(1)=1/s , L(3)=3/s , L(-3)=-3/s, ……..etc

(ii) Laplace Transform of algebraic function:

Let f(t)=t^n , n=0,1,2,…..

Then Laplace transform of f(t) is given by

L{f(t)}=L(t^n)= $\frac{n!}{s^{n+1}}$, where n! is factorial of n *

n!=n(n-1)(n-2)…..2.1

i. e. 5!=5.4.3.2.1=120 , 4!=4.3.2.1=24 ………etc

Ex: L(t)=1/s^2 , L(t^2)=2/s^3 ,………

(iii) Laplace Transform of exponential function:

Let f(t)=e^{at} ,where a is constant

Then Laplace transform of f(t) is given by

L{f(t)}=L(e^{at})= $\frac{1}{s-a}$ *

Similarly, L(e^{-at})= $\frac{1}{s+a}$ *

Ex: $L(e^t) = \frac{1}{s-1},\ L(e^{-t}) = \frac{1}{s+1}, L(e^{-2t}) = \frac{1}{s+2}, ...etc$

(iv) Laplace Transform of trigonometric function:

Let f(t)=sin at

Then Laplace transform of f(t) is given by

$L(\sin at) = \frac{a}{s^2+a^2}, \text{where a is constant}$ *

Ex: $L(\sin t) = \frac{1}{s^2+1}, L(\sin 2t) = \frac{2}{s^2+4}, L(\sin 5t) = \frac{5}{s^2+25}$ ……..etc

Let f(t)=cos at

Then Laplace transform of f(t) is given by

$L(\cos at) = \frac{s}{s^2+a^2}, \text{where a is constant}$ *

Ex: $L(\cos t) = \frac{s}{s^2+1}, L(\cos 2t) = \frac{s}{s^2+4}, L(\cos 3t) = \frac{s}{s^2+9}.........etc$

(v) Laplace Transform of hyperbolic function:

Let f(t)=sinh at (reads as sine hyperbolic)

Then Laplace transform of f(t) is given by

$$L(\sinh at) = \frac{a}{s^2 - a^2}, \text{where a is constant} \qquad *$$

Ex: $L(\sinh t) = \frac{1}{s^2 - 1}, L(\sinh 2t) = \frac{2}{s^2 - 4}, L(\sinh 5t) = \frac{5}{s^2 - 25}$ ....etc

Let f(t)=cosh at (reads as cosine hyperbolic)

Then Laplace transform of f(t) is given by

$$L(\cosh at) = \frac{s}{s^2 - a^2}, \text{where a is constant} \qquad *$$

Ex : $L(\cosh t) = \frac{s}{s^2 - 1}, L(\cosh 3t) = \frac{s}{s^2 - 9}, L(\cosh 4t) = \frac{s}{s^2 - 16}..........etc$

First Shifting Theorem:

$$If \quad L\{f(t)\} = \bar{f}(s), \text{then} \quad L\{e^{at} f(t)\} = \bar{f}(s-a), s-a > 0$$

Proof: From the def^n, we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \bar{f}(s)$$

$$\Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-st}.e^{at} f(t) dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt = \bar{f}(s-a) \quad (\text{comparing def}^n)$$

Hence Proved

Similarly

$$If \quad L\{f(t)\} = \bar{f}(s), \text{then} \quad L\{e^{-at} f(t)\} = \bar{f}(s+a),$$

Using first shifting theorem we obtain the following result.

(i) $\quad L(e^{at}) = \frac{1}{s-a} \left(\because L(1) = \frac{1}{s} \right)$

(ii) $\quad L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}, n = 0,1,2,.........$

(iii) $\quad L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$

(iv) $\quad L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$

(v) $\quad L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$

(vi) $\quad L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$

Some trigonometric formula :

(i) $\mathrm{Sin}(A+B)=\mathrm{Sin}\,A.\mathrm{Cos}\,B+\mathrm{Cos}\,A.\mathrm{Sin}\,B$

(ii) $\mathrm{Sin}(A-B)=\mathrm{Sin}\,A.\mathrm{Cos}\,B-\mathrm{Cos}\,A.\mathrm{Sin}\,B$

(iii) $\mathrm{Cos}(A+B)=\mathrm{Cos}\,A.\mathrm{Cos}\,B-\mathrm{Sin}A.\mathrm{Sin}\,B$

(iv) $\mathrm{Cos}(A-B)=\mathrm{Cos}\,A.\mathrm{Cos}\,B+\mathrm{Sin}A.\mathrm{Sin}\,B$

(v) $\mathrm{Sin}(A+B)+\mathrm{Sin}(A-B)=2\,\mathrm{Sin}\,A.\mathrm{Cos}\,B$

(vi) $\mathrm{Sin}(A+B)-\mathrm{Sin}(A-B)=2\,\mathrm{Cos}\;A.\mathrm{Sin}\,B$

(vii) $\mathrm{Cos}(A+B)+\mathrm{Cos}(A-B)=2\,\mathrm{Cos}\,A.\mathrm{Cos}\,B$

(viii) $\mathrm{Cos}(A-B)-\mathrm{Cos}(A+B)=2\,\mathrm{Sin}\,A.\mathrm{Sin}\,B$

(ix) $\mathrm{Sin}\,2A=2\,\mathrm{Sin}\,A.\mathrm{Cos}\,A$

(x) $\mathrm{Cos}\,2A=\mathrm{Cos}^2A-\mathrm{Sin}^2A=1-2\,\mathrm{Sin}^2A=2\,\mathrm{Cos}^2A-1$

(xi) $\tan^{-1}x+\cot^{-1}x=\pi/2$

(xii) $\mathrm{Sin}^2A=\dfrac{1-\mathrm{Cos}\,2A}{2}$, $\mathrm{Sin}^2 2A=\dfrac{1-\mathrm{Cos}\,4A}{2}$,.....

$(xiii)$ $\mathrm{Cos}^2A=\dfrac{1+\mathrm{Cos}\,2A}{2}$, $\mathrm{Cos}^2 2A=\dfrac{1+\mathrm{Cos}\,4A}{2}$,....

(xiv) $\mathrm{Sin}^3A=\dfrac{3\,\mathrm{Sin}\,A-\mathrm{Sin}\,3A}{4}$, $\mathrm{Cos}^3A=\dfrac{3\,\mathrm{Cos}\,A+\mathrm{Cos}\,3A}{4}$

Some Derivative formula:

(i) $\dfrac{d}{dx}(\sin x)=\cos x$, $\dfrac{d}{dx}(\cos x)=-\sin x$, $\dfrac{d}{dx}(\sin 2x)=2\cos 2x$, $\dfrac{d}{dx}(\cos 3x)=-3\sin 3x$,..........

(ii) $\dfrac{d}{dx}(uv)=u.\dfrac{dv}{dx}+v.\dfrac{du}{dx}$ $(product\ rule)$

(iii) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v.\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}$ $(division\ rule)$

(iv) $\dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}$, $\dfrac{d}{dx}(\cot^{-1}x)=-\dfrac{1}{1+x^2}$

Some Integration formula:

(i) $\int \sin x\,dx=-\cos x+c$, $\int \cos x\,dx=\sin x+c$

(ii) $\int \sin 2x\,dx=-\dfrac{\cos 2x}{2}+c$, $\int \cos 3x\,dx=\dfrac{\sin 3x}{3}+c$,.......

(iii) $\int \dfrac{1}{1+x^2}dx=\tan^{-1}x+c$, $\int \dfrac{1}{x^2+a^2}dx=\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$

Algebraic Properties of Laplace transformation:

(i) $L(u\pm v\pm w\pm........)=L(u)\pm L(v)\pm....$

(ii) $L(k.f(t))=k.L(f(t))$, where k is constant

Ex 1 : *Find Laplace transforms of the following functions:*

(i) $1+t^2-3t$, (ii) $(1+t)^2$, (iii) cos t -3sin t, (iv) $\sin^2 t$, (v) $\cos^2 t$
(vi) $\sin^2 2t$, (vii) $\cos^2 3t$, (viii) (cos t - sin t $)^2$, (ix) $1+e^{-t}$, (x) sin 3t. cos 2t
(xi) cos 4t.cos t ,(xii) sin 3t.sin t, (xiii) cos(at+b), (xiv) cosh 2t- sinh t
(xv) $\sin^3 t$, (xvi) $\cos^3 t$, (xvii) $\sin^3 2t$, (xviii) $\cos^3 3t$.

Sol^n :

$(i)\quad L(1+t^2-3t)=L(1)+L(t^2)-L(3t)=\frac{1}{s}+\frac{2!}{s^{2+1}}-3.\frac{1}{s^2}=\frac{1}{s}+\frac{2}{s^3}-\frac{3}{s^2}$

$(ii)\quad L(1+t)^2=L(1+2t+t^2)=L(1)+L(2t)+L(t^2)=\frac{1}{s}+\frac{2}{s^2}+\frac{2}{s^3}$

$(iii)\quad L(\cos t-3\sin t)=L(\cos t)-L(3\sin t)=\frac{s}{s^2+1}-\frac{3}{s^2+1}$

$(iv)\quad L(\sin^2 t)=L\left(\frac{1-\cos 2t}{2}\right)=\frac{1}{2}\{L(1)-L(\cos 2t)\}=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$

$(v)\quad L(\cos^2 t)=L\left(\frac{1+\cos 2t}{2}\right)=\frac{1}{2}\{L(1)+L(\cos 2t)\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$

$(vi)\quad L(\sin^2 2t)=L\left(\frac{1-\cos 4t}{2}\right)=\frac{1}{2}\{L(1)-L(\cos 4t)\}=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+16}\right)$

$(vii)\quad L(\cos^2 3t)=L\left(\frac{1+\cos 6t}{2}\right)=\frac{1}{2}\{L(1)+L(\cos 6t)\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+36}\right)$

$(viii)\quad L(\cos t+\sin t)^2=L(\cos^2 t+\sin^2 t+2\sin t.\cos t)$

$$=L(1+\sin 2t)=L(1)+L(\sin 2t)=\frac{1}{s}+\frac{2}{s^2+4}$$

$(ix)\quad L(1+e^{-t})=L(1)+L(e^{-t})=\frac{1}{s}+\frac{1}{s+1}$

$(x)\quad L(\sin 3t.\cos 2t)=\frac{1}{2}L(2\sin 3t.\cos 2t)$

$$=\frac{1}{2}L\{\sin(3t+2t)+\sin(3t-2t)\}$$

$$=\frac{1}{2}L(\sin 5t+\sin t)=\frac{1}{2}\left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)$$

$(xi)\quad L(\cos 4t.\cos t)=\frac{1}{2}L(2\cos 4t.\cos t)$

$$=\frac{1}{2}L(\cos(4t+t)+\cos(4t-t))$$

$$=\frac{1}{2}L(\cos 5t+\cos 3t)=\frac{1}{2}\left(\frac{s}{s^2+25}+\frac{s}{s^2+9}\right)$$

$(xii)\quad L(\sin 3t.\sin t)=\frac{1}{2}L(2\sin 3t.\sin t)$

$$=\frac{1}{2}L\{\cos(3t-t)-\cos(3t+t)\}$$

$$=\frac{1}{2}L(\cos 2t-\cos 4t)=\frac{1}{2}\left(\frac{s}{s^2+4}-\frac{s}{s^2+16}\right)$$

$(xiii)\quad L\{\cos(at+b)\}=L(\cos at.\cos b-\sin at.\sin b)$

$$=L(\cos at.\cos b)-L(\sin at.\sin b)$$

$$=\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}$$

$(xiv)\quad L\{\cosh 2t-\sinh t\}=L(\cosh 2t)-L(\sinh t)=\frac{s}{s^2-4}-\frac{1}{s^2-1}$

$(xv)\quad L(\sin^3 t)=L\left(\frac{3\sin t-\sin 3t}{4}\right)=\frac{1}{4}\left(\frac{3}{s^2+1}-\frac{3}{s^2+9}\right)$

$(xvi)\quad L(\cos^3 t)=L\left(\frac{3\cos t+\cos 3t}{4}\right)=\frac{1}{4}\left(\frac{3s}{s^2+1}+\frac{s}{s^2+9}\right)$

$(xvii)\quad L(\sin^3 2t)=L\left(\frac{3\sin 2t-\sin 6t}{4}\right)=\frac{1}{4}\left(3.\frac{2}{s^2+4}-\frac{6}{s^2+36}\right)$

$$=\frac{1}{4}\left(\frac{6}{s^2+4}-\frac{6}{s^2+36}\right)$$

$(xviii)\quad L(\cos^3 3t)=L\left(\frac{3\cos 3t+\cos 9t}{4}\right)=\frac{1}{4}\left(\frac{3s}{s^2+9}+\frac{s}{s^2+81}\right)$

Exercises-1

Find Laplace transform of the following functions:

(i) $2-3t+e^{2t}$
(ii) $(1-t^2)^2$
(iii) $(e^t+1)^2$
(iv) $(\sin t-\cos t)^2$
(v) $\text{Sin}^2 3t$
(vi) $\text{Cos}^2 2t$
(vii) Sin4t.cos2t
(viii) Cos3t.cos2t
(ix) Sin3t.sin2t
(x) Sin(at+b)
(xi) $\text{Sin}^3 3t$

Ex 2: Find Laplace transform of the following functions

(i) te^{-t} (ii) t^2e^{2t} (iii) $e^{2t}t^5$ (iv) $e^{-t}\sin t$ (v) $e^{2t}\cos 2t$ (vi) $e^{3t}\sin^2 t$
(vii) $e^{-2t}\cos^2 t$ (viii) $e^{-4t}\sin 2t \cos t$ (ix) $e^{t}\cos 3t \cos t$ (x) $e^{-t}(\cosh 2t-\sinh t)$
(xi) $(1+t)e^{-t}$

Sol^n:

(i) We have $L(t)=\frac{1}{s^2}$

Using first shifting rule, $L(te^{-t})=\frac{1}{(s+1)^2}$ $[i.e.\ s\to s-(-1)=s+1]$

(ii) We have $L(t^2)=\frac{2!}{s^{2+1}}=\frac{2}{s^3}$

Using first shifting rule, $L(t^2e^{2t})=\frac{2}{(s-2)^3}$

(iii) We have $L(t^5)=\frac{5!}{s^{5+1}}=\frac{120}{s^6}$

Using first shifting rule, $L(e^{2t}t^5)=\frac{120}{(s-2)^6}$

(iv) We have $L(\sin t)=\frac{1}{s^2+1}$

Using first shifting rule, $L(e^{-t}\sin t)=\frac{1}{(s+1)^2+1}=\frac{1}{s^2+2s+2}$

(v) We have $L(\cos 2t)=\frac{s}{s^2+4}$

Using first shifting rule, $L(e^{2t}\cos 2t)=\frac{s-2}{(s-2)^2+4}=\frac{s-2}{s^2-4s+8}$

(vi) We have $L(\sin^2 t)=L\left(\frac{1-\cos 2t}{2}\right)=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$

Using first shifting rule, $L(e^{3t}\sin^2 t)=\frac{1}{2}\left(\frac{1}{s-3}-\frac{s-3}{(s-3)^2+4}\right)$

(vii) We have $L(\cos^2 t)=L\left(\frac{1+\cos 2t}{2}\right)=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$

Using first shifting rule, $L(e^{-2t}\cos^2 t)=\frac{1}{2}\left(\frac{1}{s+2}+\frac{s+2}{(s+2)^2+4}\right)$

$(viii)$ We have $L(\sin 2t \cos t) = \frac{1}{2} L(2\sin 2t \cos t)$

$$= \frac{1}{2} L(\sin 3t + \sin t) = \frac{1}{2}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)$$

Using first shifting rule, $L(e^{-4t} \sin 2t \cos t) = \frac{1}{2}\left(\frac{3}{(s+4)^2+9} + \frac{1}{(s+4)^2+1}\right)$

(ix) We have $L(\cos 3t \cos t) = \frac{1}{2} L(2\cos 3t \cos t)$

$$= \frac{1}{2} L(\cos 4t + \cos 2t) = \frac{1}{2}\left(\frac{s}{s^2+16} + \frac{s}{s^2+4}\right)$$

Using first shifting rule, $L(e^{t} \cos 3t \cos t) = \frac{1}{2}\left(\frac{s-1}{(s-1)^2+16} + \frac{s-1}{(s-1)^2+4}\right)$

(x) We have $L(\cosh 2t - \sinh t) = \left(\frac{s}{s^2-4} - \frac{1}{s^2-1}\right)$

Using first shifting rule, $L\{e^{-t}(\cosh 2t - \sinh t)\} = \frac{s+1}{(s+1)^2-4} - \frac{1}{(s+1)^2-1}$

(xi) We have $L(1+t) = \frac{1}{s} + \frac{1}{s^2}$

Using first shifting rule, $L\{(1+t)e^{-t}\} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$

Exercises-2

Find Laplace transform of the following functions:

(i) te^{2t} (ii) t^3e^{-t} (iii) $e^{-2t}\sin 3t$ (iv) $e^{t}\cos 4t$
(v) $e^{2t}\cos^2 2t$ (vi) $e^{-t}\sin^2 3t$ (vii) $e^{-t}\sin^3 t$
(viii) $e^{t}\cos^3 t$ (ix) $e^{-t}(1-t^2)$ (x) $e^{2t}\sinh t$
(xi) $e^{t}(\cosh 2t - \sin t)$
(xii) $e^{2t}(3t^5 - \cos 4t)$
(xiii) $e^{-t}\sin 4t \cos 2t$
(xiv) $e^{-3t}\sin 5t \sin 3t$

Change of scale property:

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Laplace Transforms of Derivatives:

If $L\{f(t)\} = \bar{f}(s)$ and $f'(t)$ is contineous, then

$L\{f'(t)\} = s\,\bar{f}(s) - f(0)$ *

$L\{f''(t)\} = s^2\,\bar{f}(s) - sf(0) - f'(0)$ *

$L\{f'''(t)\} = s^3\,\bar{f}(s) - s^2 f(0) - sf'(0) - f''(0)$ *

and so on

Laplace Transforms of Integrals:

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$ *

Laplace Transforms of t^n f(t) :(Multiplication by t , t^2,.....)

If $L\{f(t)\} = \bar{f}(s)$, then for a positive integer n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left\{\bar{f}(s)\right\}$$

Particularly, for n = 1, $L\{t\,f(t)\} = (-1)^1 \frac{d^1}{ds^1}\left\{\bar{f}(s)\right\} = -\frac{d}{ds}\left\{\bar{f}(s)\right\}$ *

for n = 2, $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\left\{\bar{f}(s)\right\} = \frac{d^2}{ds^2}\left\{\bar{f}(s)\right\}$ *

Laplace transform of $\frac{f(t)}{t}$ (division by t)

If $L\{f(t)\} = \bar{f}(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)\,ds$ *

Ex 3 : Find the Laplace transforms of the following functions :

(i) t sint (ii) t cost (iii) t^2sint (iv) t^2 cost

(v) t $\sin^2$t (vi) t cos2t (vii) t e^{-t}sin4t (viii) t sin3t.cos2t

(ix) t e^{-t} cosht (x) t^2 sinat

Sol^n :

(i) Here $f(t) = \sin t$

$\therefore L(f(t)) = L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s)$

So, by using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{0-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2} \quad (by\ \ division\ rule)$$

(ii) Here $f(t) = \cos t$

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t\, f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cos t) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left(\frac{1.(s^2+1) - s.2s}{(s^2+1)^2}\right) = \frac{s^2-1}{(s^2+1)^2}$$

(iii) Here $f(t) = \sin t$

$$\therefore L(f(t)) = L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t^2 f(t)) = \frac{d^2}{ds^2}\left(\bar{f}(s)\right)$, we get

$$L(t^2 \sin t) = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right) = \frac{2.(s^2+1)^2 - 2s.2(s^2+1).2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)\{2(s^2+1) - 8s^2\}}{(s^2+1)^4} = \frac{2-6s^2}{(s^2+1)^3}$$

(iv) Here $f(t) = \cos t$

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} = \bar{f}(s)$$

so, by using result $L(t^2 f(t)) = \frac{d^2}{ds^2}\left(\bar{f}(s)\right)$, we get

$$L(t^2 \cos t) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{1.s^2+1-s.2s}{(s^2+1)^2}\right) = \frac{d}{ds}\frac{1-s^2}{(s^2+1)^2}$$

$$= \frac{-2s.(s^2+1)^2 - (1-s^2).2(s^2+1).2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)(-2s^3 - 2s - 4s + 4s^3)}{(s^2+1)^4} = \frac{2s^3-6s}{(s^2+1)^3}$$

(v) Here $f(t)=\sin^2 t$

$$\therefore L(\sin^2 t)=L\left(\frac{1-\cos 2t}{2}\right)=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)=\bar{f}(s)$$

So, by using result $L(t\,f(t))=-\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\sin^2 t)=-\frac{d}{ds}\left\{\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right\}=-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$=-\frac{1}{2}\left(-\frac{1}{s^2}-\frac{1.s^2+4-s.2s}{(s^2+4)^2}\right)=\frac{1}{2s^2}+\frac{4-s^2}{2(s^2+4)^2}$$

(vi) Here $f(t)=\cos 2t$

$$\therefore L(\cos 2t)=\frac{s}{s^2+4}=\bar{f}(s)$$

So, by using result $L(t\,f(t))=-\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cos 2t)=-\frac{d}{ds}\left(\frac{s}{s^2+4}\right)=-\frac{1.s^2+4-s.2s}{(s^2+4)^2}=\frac{4-s^2}{(s^2+4)^2}$$

(vii) We have $L(\sin 4t)=\frac{4}{s^2+16}$

So, by using result $L(t\,f(t))=-\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\sin 4t)=-\frac{d}{ds}\left(\frac{4}{s^2+16}\right)=-\frac{0-4.2s}{(s^2+16)^2}=\frac{8s}{(s^2+16)^2}$$

Now by using first shifting property, we get

$$L(e^{-t}\,t\sin 4t)=\frac{8(s+1)}{\left[(s+1)^2+16\right]^2}=\frac{8(s+1)}{(s^2+2s+17)^2}$$

(ix) We have $L(\cosh t)=\frac{s}{s^2-1}$

So, by using result $L(t\,f(t))=-\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t\cosh t)=-\frac{d}{ds}\left(\frac{s}{s^2-1}\right)=-\frac{1.s^2-1-s.2s}{(s^2-1)^2}=\frac{s^2+1}{(s^2-1)^2}$$

Now by using first shifting property, we get

$$L(e^{-t}\,t\cosh t)=\frac{(s+1)^2+1}{\left[(s+1)^2-1\right]^2}=\frac{s^2+2s+2}{(s^2+2s)^2}$$

$(viii)$ Here $f(t) = \sin 3t.\cos 2t$

$$\therefore L(\sin 3t.\cos 2t) = \frac{1}{2}L(2\sin 3t.\cos 2t)$$

$$= \frac{1}{2}L(\sin 5t + \sin t) = \frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right) = \bar{f}(s)$$

So, using result $L(t\,f(t)) = -\frac{d}{ds}\left(\bar{f}(s)\right)$, we get

$$L(t.\sin 3t \cos 2t) = -\frac{d}{ds}\frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right) = -\frac{1}{2}\left(\frac{0-5.2s}{(s^2+25)^2} + \frac{0-2s}{(s^2+1)^2}\right)$$

$$= -\frac{1}{2}\left(\frac{-10s}{(s^2+25)^2} + \frac{-2s}{(s^2+1)^2}\right) = \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

(x) Here $f(t) = \sin at$

$$\therefore L(\sin at) = \frac{a}{s^2+a^2} = \bar{f}(s)$$

So by using result $L(t^2\,f(t)) = \frac{d^2}{ds^2}\left(\bar{f}(s)\right)$, we get

$$L(t^2 \sin at) = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{0-a.2s}{(s^2+a^2)^2}\right) = -2a\frac{d}{ds}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$= -2a\left(\frac{1.(s^2+a^2)^2 - s.2(s^2+a^2).2s}{(s^2+a^2)^4}\right)$$

$$= \frac{-2a(s^2+a^2)\{s^2+a^2-4s^2\}}{(s^2+a^2)^4} = \frac{-2a(a^2-3s^2)}{(s^2+a^2)^3}$$

Exercises-3

Find Laplace transforms of the following functions:

(i) t sin2t (ii) t cos 3t (iii) t^2 cos2t (iv)t e^t sin2t

(v) t e^{-2t} cos t (vi) te^{-t} sinh 2t (vii)t cos4t cos t (viii)t^2 e^{-t} cos t

Ex 4 : Find the Laplace transforms of the following functions :

$(i)\ \frac{1-e^t}{t}$, $(ii)\ \frac{e^{-t}-1}{t}$, $(iii)\ \frac{e^t - e^{-t}}{t}$, $(iv)\ \frac{e^{-at}-e^{-bt}}{t}$

$(v)\ \frac{1-\cos 2t}{t}$, $(vi)\ \frac{\cos 2t - \cos 3t}{t}$, $(vii)\ \frac{\sin t}{t}$ $(viii)\ \frac{\sin^2 t}{t}$

$(ix)\ \frac{e^{-t}\sin t}{t}$, $(x)\ \frac{\sin at}{t}$, $(xi)\ \frac{e^{at} - \cos bt}{t}$

Sol^n:

(i) Here $f(t) = 1 - e^t$

$$\bar{f}(s) = L(1 - e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$\therefore L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= [\log s - \log(s-1)]_s^\infty = \left[\log\frac{s}{s-1}\right]_s^\infty = \log\frac{s-1}{s}$$

(ii) Here $f(t) = e^{-t} - 1$

$$\bar{f}(s) = L(e^{-t} - 1) = \frac{1}{s+1} - \frac{1}{s}$$

$$\therefore L\left(\frac{e^{-t}-1}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s}\right) ds$$

$$= [\log(s+1) - \log s]_s^\infty = \left[\log\frac{s+1}{s}\right]_s^\infty = \log\frac{s}{s+1}$$

(iii) Here $f(t) = e^t - e^{-t}$

$$\bar{f}(s) = L(e^t - e^{-t}) = \frac{1}{s-1} - \frac{1}{s+1}$$

$$\therefore L\left(\frac{e^t - e^{-t}}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s+1}\right) ds$$

$$= [\log(s-1) - \log(s+1)]_s^\infty = \left[\log\frac{s-1}{s+1}\right]_s^\infty = \log\frac{s+1}{s-1}$$

(iv) Here $f(t) = e^{-at} - e^{-bt}$

$$\bar{f}(s) = L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\therefore L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty = \left[\log\frac{s+a}{s+b}\right]_s^\infty = \log\frac{s+b}{s+a}$$

(v) Here $f(t) = 1 - \cos 2t$

$$\bar{f}(s) = L(1 - \cos 2t) = \frac{1}{s} - \frac{s}{s^2 + 4}$$

$$\therefore L\left(\frac{1 - \cos 2t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2 + 4\right)\right]_s^\infty = \left[\log s - \log\sqrt{s^2 + 4}\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2 + 4}}\right]_s^\infty = \log \frac{\sqrt{s^2 + 4}}{s}$$

(vi) Here $f(t) = \cos 2t - \cos 3t$

$$\bar{f}(s) = L(\cos 2t - \cos 3t) = \frac{s}{s^2 + 4} - \frac{s}{s^2 + 9}$$

$$\therefore L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9}\right) ds$$

$$= \left[\frac{1}{2}\log\left(s^2 + 4\right) - \frac{1}{2}\log\left(s^2 + 9\right)\right]_s^\infty$$

$$= \frac{1}{2}\log\left[\frac{s^2 + 4}{s^2 + 9}\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2 + 9}{s^2 + 4}\right)$$

(vii) Here $f(t) = \sin t$

$$\bar{f}(s) = L(\sin t) = \frac{1}{s^2 + 1}$$

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2 + 1}\,ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$(viii)$ Here $f(t) = \sin^2 t$, $\bar{f}(s) = L(\sin^2 t) = L\left(\frac{1 - \cos 2t}{2}\right)$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4}\right)$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4}\right) ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2 + 4\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2 + 4}\right]_s^\infty = \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2 + 4}}\right]_s^\infty = \frac{1}{2}\log\left(\frac{\sqrt{s^2 + 4}}{s}\right)$$

(ix) Here $f(t)=e^{-t}\sin t$

$$\bar{f}(s)=L(e^{-t}\sin t)=\frac{1}{(s+1)^2+1} \quad \text{(by shifting rule)}$$

$$\therefore L\left(\frac{e^{-t}\sin t}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$=\left[\tan^{-1}(s+1)\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}(s+1)$$

$$=\frac{\pi}{2}-\tan^{-1}(s+1)=\cot^{-1}(s+1)$$

(x) Here $f(t)=\sin at$

$$\bar{f}(s)=L(\sin at)=\frac{a}{s^2+a^2}$$

$$\therefore L\left(\frac{\sin at}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty \frac{a}{s^2+a^2}ds=a.\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$=\left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}\left(\frac{s}{a}\right)$$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a}\right)=\cot^{-1}\left(\frac{s}{a}\right)$$

(xi) Here $f(t)=e^{at}-\cos bt$

$$\bar{f}(s)=L(e^{at}-\cos bt)=\frac{1}{s-a}-\frac{s}{s^2+b^2}$$

$$\therefore L\left(\frac{e^{at}-\cos bt}{t}\right)=\int_s^\infty \bar{f}(s)ds=\int_s^\infty\left(\frac{1}{s-a}-\frac{s}{s^2+b^2}\right)ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty=\left[\log(s-a)-\log\sqrt{s^2+b^2}\right]_s^\infty$$

$$=\left[\log\frac{s-a}{\sqrt{s^2+b^2}}\right]_s^\infty=\log\frac{\sqrt{s^2+b^2}}{s-a}$$

<u>Exercises-4</u>

<u>Find Laplace transforms of the following functions</u>:

$(i)\ \frac{1-e^{at}}{t}$ $\quad (ii)\ \frac{e^{at}-e^{bt}}{t}$ $\quad (iii)\ \frac{1-\cos t}{t}$ $\quad (iv)\ \frac{\cos at-\cos bt}{t}$

$(v)\ \frac{\sin 2t}{t}$ $\quad (vi)\ \frac{e^{t}\sin t}{t}$ $\quad (vii)\ \frac{e^{-t}-e^{-3t}}{t}$

Ex 5 : Find the Laplace transforms of the following functions :

$(i)\int_0^t e^{-t}\cos t\,dt \qquad (ii)\int_0^t \frac{\sin t}{t}dt \quad (iii)\int_0^t e^{t}\left(\frac{\sin t}{t}\right)dt$

$(iv)\int_0^t \frac{\cos at-\cos bt}{t}dt$

Sol^n:

(i) Here $\mathrm{f(t)}=e^{-t}\cos t$

$$\mathrm{L\{f(t)\}}=\mathrm{L}\left(e^{-t}\cos t\right)=\frac{s+1}{(s+1)^2+1}=\frac{s+1}{s^2+2s+2}=\bar{\mathrm{f}}(s)$$

$$\therefore \mathrm{L}\left(\int_0^t e^{-t}\cos t\,dt\right)=\frac{\bar{\mathrm{f}}(s)}{s}=\frac{1}{s}.\frac{s+1}{s^2+2s+2}=\frac{s+1}{s\left(s^2+2s+2\right)}$$

(ii) Here $\mathrm{f(t)}=\frac{\sin t}{t}$

$$\mathrm{L}\left(\frac{\sin t}{t}\right)=\int_s^\infty \bar{\mathrm{f}}(s)ds=\int_s^\infty \mathrm{L(\sin t)}ds \qquad (by\ division\ rule)$$

$$=\int_s^\infty \frac{1}{s^2+1}ds=\left[\tan^{-1}s\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s$$

$$=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s=\bar{\mathrm{f}}(s)$$

$$\therefore \mathrm{L}\left(\int_0^t \frac{\sin t}{t}dt\right)=\frac{\bar{\mathrm{f}}(s)}{s}=\frac{\cot^{-1}s}{s}$$

(iii) Here $\mathrm{f(t)}=\left(e^{t}\left(\frac{\sin t}{t}\right)\right)$

$$\text{Now }\mathrm{L}\left(\frac{\sin t}{t}\right)=\int_s^\infty \bar{\mathrm{f}}(s)ds=\int_s^\infty \mathrm{L(\sin t)}ds \qquad (by\ division\ rule)$$

$$=\int_s^\infty \frac{1}{s^2+1}ds=\left[\tan^{-1}s\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s$$

$$=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$$

$$\text{So, }\mathrm{L}\left(e^{t}\left(\frac{\sin t}{t}\right)\right)=\cot^{-1}(s-1)=\bar{\mathrm{f}}(s) \quad (by\ first\ shifting\ rule)$$

$$\therefore \mathrm{L}\left(\int_0^t e^{t}\left(\frac{\sin t}{t}\right)dt\right)=\frac{\bar{\mathrm{f}}(s)}{s}=\frac{\cot^{-1}(s-1)}{s}$$

(iv) Here $f(t) = \frac{\cos at - \cos bt}{t}$

Now $L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \bar{f}(s)ds = \int_s^\infty L(\cos at - \cos bt)ds$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) = \bar{f}(s)$$

$$\therefore L\left(\int_0^t \frac{\cos at - \cos bt}{t}dt\right) = \frac{\bar{f}(s)}{s} = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Exercise-5

Find Laplace transforms of the following functions:

$(i)\ \int_0^t e^{2t}\sin t\,dt \quad (ii)\ \int_0^t \sin 2t\cos t\,dt \quad (iii)\ \int_0^t t\sin 2t\,dt$

$(iv)\ \int_0^t t\,e^{-t}\sin 4t\,dt \quad (v)\ \int_0^t \frac{\cos 2t - \cos 3t}{t}dt$

(vi) By using the Laplace transform of sin at, find Laplace transform of cos at.

Ex 6: By using the Laplace transform of cos at, find Laplace transform of sin at.

Sol^n : Let $f(t) = \cos at$. Then $L(f(t)) = \frac{s}{s^2+a^2} = \bar{f}(s)$

Now $f'(t) = -a\sin at$

$$\Rightarrow L(f'(t)) = -a\,L(\sin at)$$

$$\Rightarrow L(\sin at) = -\frac{1}{a}L(f'(t))$$

$$= -\frac{1}{a}\left(s\bar{f}(s) - f(0)\right) \quad (\textit{by Laplace transform of derivative})$$

$$= -\frac{1}{a}\left(s.\frac{s}{s^2+a^2} - 1\right) = -\frac{1}{a}\left(\frac{s^2-s^2-a^2}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2}$$