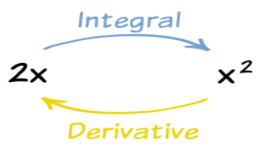
CHAPTER-1

Integration is an inverse process of differentiation or antiderivative process.

If the derivative of $F(x)$ is $f(x)$, then the antiderivative or integral of $f(x)$ is $F(x)$.

Let $\frac{d}{dx}\left(F(x)\right)=f(x) \Rightarrow$ integration of $f(x)=F(x)$



Again $\frac{d}{dx}\left(F(x)+c\right)=f(x) \Rightarrow \int f(x)\,dx=F(x)+c$

Where c is called constant of integration.

Here $f(x)$ is called integrand and $F(x)$ is called integral.

But dx represents integration with respect x.

The symbol $\int$ represents sign of integration.

Ex: We have $\frac{d}{dx}(\sin x)=\cos x \Rightarrow \int \cos x\,dx=\sin x+c$

FORMULAS :

$F-1 \int x^n dx=\frac{x^{n+1}}{n+1}+c\,(n\neq -1)$

$Ex: Evaluate \int x^9 dx$

$Ans: \int x^9 dx=\frac{x^{9+1}}{9+1}+c=\frac{x^{10}}{10}+c$

$Ex: Evaluate \int x^{\frac{5}{2}} dx$

$Ans: \int x^{\frac{5}{2}} dx=\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+c=\frac{2}{7}x^{\frac{7}{2}}+c$

$Ex: Evaluate \int x^{-\frac{5}{2}} dx$

$Ans: \int x^{-\frac{5}{2}} dx=\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}+c=-\frac{2}{3}x^{-\frac{3}{2}}+c$

$Ex: Evaluate \int \frac{1}{x^7} dx$

$Ans: \int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-6}}{-6} + c = -\frac{x^{-6}}{6} + c$

$Ex: Evaluate \int \frac{1}{x\sqrt{x}} dx$

$Ans: \int \frac{1}{x\sqrt{x}} dx = \int \frac{1}{x^{\frac{3}{2}}} dx = \int x^{-\frac{3}{2}} dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -2x^{-\frac{1}{2}} + c$

(IMP)

$F-2 \int e^x dx = e^x + c$

$F-3 \int a^x dx = \frac{a^x}{\ln a} + c$

$Ex: Ealuate \int 3^x dx$

$Ans: \int 3^x dx = \frac{3^x}{\ln 3} + c$

$F-4 \int \sin x\, dx = -\cos x + c$

$F-5 \int \cos x\, dx = \sin x + c$

$F-6 \int \sec^2 x\, dx = \tan x + c$

$F-7 \int \sec x \tan x\, dx = \sec x + c$

$F-9 \int \cos ec^2 x\, dx = -\cot x + c$

$F-10 \int \cos ecx \cot x\, dx = -\cos ecx + c$

$F-11 \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$F-12 \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$F-13 \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

$F-14 \int \frac{1}{x} dx = \ln|x| + c$

$Note: \int x dx = \frac{x^2}{2} + c$

$\int dx = x + c$

Algebra of integration:

F-1: $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

Ex: $\int (sinx + cosx - e^x - sec^2x)\, dx$

$$= \int sinxdx + \int cosxdx - \int e^x dx - \int sec^2xdx$$

$= -cosx + sinx - e^x - tanx + c$

F-2: $\int kf(x)dx = k\int f(x)dx,$ *where k is a constant*.

Ex: $\int 5x^9dx = 5\frac{x^{10}}{10} + c = \frac{x^{10}}{2} + c$

Ex: $\int 5dx = 5x + c$

Ex: $\int \frac{7}{x}dx = 7ln|x| + c$

Ex: $\int 3sinxdx = -3cosx + c$

Ex: $\int \frac{5}{\sqrt{1-x^2}}dx = 5sin^{-1}x + c$

Ex: $\int \frac{7}{1+x^2}dx = 7tan^{-1}x + c$

Ex: $\int (3sinx - 4cosec^2x + 9sec^2x - 5e^x - 5)dx$

$$= \int 3sinxdx - \int 4cosec^2xdx + \int 9sec^2xdx - \int 5e^xdx - \int 5dx$$

$= -3cosx + 4cotx + 9tanx - 5e^x - 5x + c$

F-3: $\frac{d}{dx}\{\int f(x)dx\} = f(x)$,The differentiation of an integral is the integrand itself.

Ex : $\frac{d}{dx}\{\int sinxdx\} = sinx$

Ex: $\int \frac{1}{cos^2x}dx = \int sec^2xdx = tanx + c$

Ex:$\int \frac{1}{sin^2x} dx = \int cosec^2xdx = -cotx + c$

Ex:$\int \frac{1}{1-sin^2x} dx = \int \frac{1}{cos^2x} dx = \int sec^2xdx = tanx + c$

Ex:$\int \frac{1}{1-cos^2x} dx = \int \frac{1}{sin^2x} dx = \int cosec^2xdx = -cotx + c$

Ex:$\int \frac{cosx}{sin^2x} dx = \int cotx\ cosecxdx = -cosecx + c$

Ex:$\int \frac{sinx}{cos^2x} dx = \int tanx\ secxdx = secx + c$

IMP

Ex:$\int tan^2xdx = \int (sec^2x - 1)dx = \int sec^2xdx - \int 1\, dx = tanx - x + c$

Ex:$\int cot^2xdx = \int (cosec^2x - 1)dx = \int cosec^2xdx - \int 1\, dx = -cotx - x + c$

Ex: $\int \frac{1-sinx}{cos^2x} dx = \int \frac{1}{cos^2x} d - \int \frac{sinx}{cos^2x} dx$

$$= \int sec^2xdx - \int tanx\ secxdx = tanx - secx + c$$

Ex:$\int \frac{1-sin^3x}{sin^2x} dx = \int \frac{1}{sin^2x} dx - \int \frac{sin^3x}{sin^2x} dx = \int cosec^2x\, dx - \int sinxdx$

$= -cotx + cosx + c$

Ex:$\int \frac{cos2x}{cos^2xsin^2x} dx = \int \frac{cos^2x-sin^2x}{cos^2xsin^2x} dx = \int \frac{cos^2x}{cos^2xsin^2x} dx - \int \frac{sin^2x}{cos^2xsin^2x} dx$

=$\int cosec^2x\, dx - \int sec^2x\, dx = -cotx - tanx + c$

Ex:$\int \frac{1}{cos^2xsin^2x} dx = \int \frac{cos^2x+sin^2x}{cos^2xsin^2x} dx = \int \frac{cos^2x}{cos^2xsin^2x} dx + \int \frac{sin^2x}{cos^2xsin^2x} dx$

=$\int cosec^2x\, dx + \int sec^2x\, dx = -cotx + tanx + c$

Ex:$\int \frac{cos2x}{cosx+sinx} dx = \int \frac{cos^2x-sin^2x}{cosx+sinx} dx = \int \frac{(cosx+sinx)(cosx-sinx)}{cosx+sinx} dx$

=$\int (cosx - sinx)dx = \int cosxdx - \int sinxdx = sinx + cosx + c$

Ex:$\int (tanx + cotx)^2dx = \int (tan^2x + cot^2x + 2tanx\ cotx)\, dx$

$=\int (sec^2x - 1 + cosec^2x - 1 + 2)\,dx = \int sec^2x dx + \int cosec^2x dx$

$= tanx - cotx + c$

Ex: $\int \frac{x^2}{1+x^2}dx = \int \frac{x^2+1-1}{1+x^2}dx = \int \frac{x^2+1}{1+x^2}dx - \int \frac{1}{1+x^2}dx = \int 1dx - \int \frac{1}{1+x^2}dx$

$= x - tan^{-1}x + c$

Ex: $\int \frac{x^4}{1+x^2}dx = \int \frac{x^4-1+1}{1+x^2}dx$

$= \int \frac{x^4-1}{1+x^2}dx + \int \frac{1}{1+x^2}dx = \int \frac{(x^2+1)(x^2-1)}{1+x^2}dx + \int \frac{1}{1+x^2}dx$

$=\int (x^2-1)\,dx + \int \frac{1}{1+x^2}dx = \int x^2dx - \int 1\,dx + \int \frac{1}{1+x^2}dx = \frac{x^3}{3} - x + tan^{-1}x + c$

Ex: $\int \frac{x^6}{1+x^2}dx = \int \frac{x^6+1-1}{1+x^2}dx = \int (\frac{(x^2)^3+1^3}{1+x^2} - \frac{1}{1+x^2})dx$

$=\int \frac{(x^2+1)(x^4-x^2+1)}{1+x^2}dx - \int \frac{1}{1+x^2}dx = \int (x^4 - x^2 + 1)\,dx - \int \frac{1}{1+x^2}dx$

$=\int x^4dx - \int x^2dx + \int dx - \int \frac{1}{1+x^2}dx = \frac{x^5}{5} - \frac{x^3}{3} + x - tan^{-1}x + c$

IMP:

Ex: $\int \frac{1}{1+sinx}dx = \int \frac{1-sinx}{(1+sinx)(1-sinx)}dx = \int \frac{1-sinx}{(1-sin^2x)}dx$

$=\int \frac{1-sinx}{cos^2x}dx = \int \frac{1}{cos^2x}d - \int \frac{sinx}{cos^2x}dx$

$= \int sec^2xdx - \int tanx\ secxdx = tanx - secx + c$

Ex: $\int \frac{sinx}{1+sinx}dx = \int \frac{sinx(1-sinx)}{(1+sinx)(1-sinx)}dx = \int \frac{sinx-sin^2x}{(1-sin^2x)}dx$

$=\int \frac{sinx-sin^2x}{cos^2x}dx = \int \frac{sinx}{cos^2x}d - \int \frac{sin^2x}{cos^2x}dx = \int tanx\ secx\ dx - \int tan^2xdx$

$= \int tanx\ secx\ dx - \int (sec^2x - 1)dx = secx - tanx + x + c$

Assignment:

Ex: $\int \frac{cosx}{1-cosx} dx$

Ex: $\int \frac{cosx}{1+cosx} dx$

Ex: $\int \frac{sinx}{1-sinx} dx$

Ex: $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

Ex: $\int \frac{x^5+5x^2-2x+7}{x^3} dx$

Ex: $\int (e^{xloga} + e^{alogx} + e^{aloga})\, dx$

Ex: $\int \frac{3-2cosx}{sin^2x} dx$

Ex: $\int \sqrt{1 - sin2x} dx$

Ex: $\int \sqrt{1 + sin2x} dx$

Ex: $\int \sqrt{1 - cos2x} dx$

Ex: $\int \sqrt{1 + cos2x} dx$

Ex: $\int \frac{cosecx}{cosecx-cotx} dx$

Ex: $\int 3^{x-2} dx$

Ex: $\int \frac{1}{1-sinx} dx$

Ex: $\int \frac{1}{1+cosx} dx$

Ex: $\int \frac{1}{1-cosx} dx$

CH-2, INTEGRATION BY SUBSTITUTIONS

Integration by substitution will be used to solve the integration easily by using suitable substitution.

If the integrand in the form of $\int \boldsymbol{f(x)f'(x)dx}$

How to solve: $\int f(x)f'(x)dx$

Let f(x) = t , take derivative in both sides

$$\Rightarrow \frac{d}{dx}f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$$

So $\int f(x)f'(x)dx = \int tdt = \frac{t^2}{2} + c = \frac{(f(x))^2}{2} + c$

Ex: Evaluate $\int sinx\ cosx\ dx$

Ans: $\int sinx\ cosx\ dx$

$let\ sinx = t$

$$\Rightarrow \frac{d}{dx}sinx = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$$

So $\int sinx\ cosx\ dx = \int t\ dt = \frac{t^2}{2} + c = \frac{sin^2x}{2} + c$

$\int \boldsymbol{(f(x))^n f'(x)dx}$

How to solve:

Let f(x) = t, take derivative in both sides

$$\Rightarrow \frac{d}{dx}f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$$

$$= \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{(f(x))^{n+1}}{n+1} + c$$

Ex: $Evaluate \int sin^5x\ cosx\ dx$

Ans: $\int sin^5x\ cosx\ dx$

$let\ sinx = t$

$\Rightarrow \frac{d}{dx}sinx = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$

So $\int sin^5x\ cosx\ dx = \int t^5dt = \frac{t^6}{6} + c = \frac{sin^6x}{6} + c$

Ex: $\int tan^3x\ sec^2x\ dx$

Ans: $\int tan^3x\ sec^2x\ dx$

$let\ tanx = t$

$\Rightarrow \frac{d}{dx}tanx = \frac{dt}{dx} \Rightarrow sec^2x\ dx = dt$

So $\int tan^3x\ sec^2x\ dx = \int t^3dt = \frac{t^4}{4} + c = \frac{tan^4x}{4} + c$

$\int \boldsymbol{f(g(x))g'(x)dx}$

How to solve:

Let g(x) = t, take derivative in both sides

$\Rightarrow \frac{d}{dx}g(x) = \frac{dt}{dx} \Rightarrow g'(x) = \frac{dt}{dx} \Rightarrow g'(x)dx = dt$

$$so \int f(g(x))g'(x)dx$$

$$= \int f(t)dt = F(t) + c = F(g(x)) + c \quad [\because \int f(x)dx = F(x) + c$$

Ex: $Evaluate \int \cos(sinx)\ cosxdx$

Ans: $\int \cos(sinx)\ cosxdx$

$let\ sinx = t \Rightarrow \frac{d}{dx}(sinx) = \frac{dt}{dx} \Rightarrow cosx\ dx = dt$

$= \int cost\ dt = sint + c = \sin(sinx) + c$

Ex:$\int e^{tanx} sec^2x\ dx$

Ans: $\int e^{tanx} sec^2x\ dx$

Let $tanx = t \Rightarrow sec^2xdx = dt$

so$\int e^{tanx} sec^2x\ dx = \int e^t dt = e^t + c = e^{tanx} + c$

Ex: $\int x^2 e^{x^3}\ dx$

Ans: $\int x^2 e^{x^3}\ dx$

Let $x^3 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2dx = dt/3$

so$\int x^2 e^{x^3}\ dx = \int \frac{e^t dt}{3} = \frac{1}{3}e^t + c = \frac{1}{3}e^{x^3} + c$

Ex:$\int a^{sinx} cosx\ dx$

Ans: $\int a^{sinx} cosx\ dx$

Let $sinx = t \Rightarrow cosxdx = dt$

$=\int a^t dt = \frac{a^t}{lna} + c$

Ex:$\int \frac{b^{lnx}}{x} dx$

Let $lnx = t \Rightarrow \frac{1}{x}dx = dt$

$=\int b^t dt = \frac{b^t}{lnb} + c$

2. $\int \frac{f'(x)}{(f(x))^n} dx$

How to solve:

Let f(x) = t, take derivative in both sides

$\Rightarrow \frac{d}{dx} f(x) = \frac{dt}{dx} \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$

So $\int \frac{f'(x)}{(f(x))^n} dx = \int \frac{dt}{t^n} = \int t^{-n} dt = \frac{t^{-n+1}}{-n+1} + c = \frac{(f(x))^{1-n}}{1-n} + c$

Ex: Evaluate $\int \frac{sec^2x}{tanx} dx$

Ans: $\int \frac{sec^2x}{tanx} dx$

Let $tanx = t \Rightarrow \frac{d}{dx} tanx = \frac{dt}{dx} \Rightarrow sec^2x dx = dt$

$$= \int \frac{dt}{t} = ln|t| + c = ln|tanx| + c$$

Ex: Evaluate $\int \frac{sec^2x}{tan^3x} dx$

Ans: $\int \frac{sec^2x}{tan^3x} dx$

Let $tanx = t \Rightarrow \frac{d}{dx} tanx = \frac{dt}{dx} \Rightarrow sec^2x dx = dt$

$= \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-2}}{-2} + c = -\frac{1}{2}(tanx)^{-2} + c$

Ex: $\int \frac{cosx}{3+4sinx} dx$

Let $3 + 4sinx = t \Rightarrow 4cosxdx = dt \Rightarrow cosxdx = \frac{dt}{4}$

$= \int \frac{dt/4}{t} = \frac{1}{4} \int \frac{dt}{t} = \frac{1}{4} ln|t| + c = \frac{1}{4} ln|3 + 4sinx| + c$

Ex: $\int \frac{x}{a^2+x^2} dx$

Let $a^2 + x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = dt/2$

$= \int \frac{dt/2}{t} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} ln|t| + c$

Ex: $\int \frac{x}{\sqrt{a^2+x^2}} dx$

Let $a^2 + x^2 = t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

$= \int \frac{tdt}{t} = \int dt = t + c = \sqrt{a^2 + x^2} + c$

Ex: $\int \frac{sec^2\sqrt{x}}{\sqrt{x}} dx$

Let $\sqrt{x}$=t $\Rightarrow \frac{1}{2\sqrt{x}}$dx=dt $\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$

$= \int sec^2t\, 2dt = 2\int sec^2t\, dt = 2tant + c = 2tan\sqrt{x} + c$

Ex: $\int \frac{(sin^{-1}x)^2}{\sqrt{1-x^2}} dx$

let $sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$= \int \frac{(sin^{-1}x)^2}{\sqrt{1-x^2}} dx = \int t^2 dt = \frac{t^3}{3} + c$

<u>Some formulas related to substitutions</u>

F-1. $\int f(ax + b)dx = \frac{1}{a} F(ax + b) + c$

Proof: Let $ax + b = t$

$\Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a}$

$so \int f(ax + b)dx = \int f(t) \frac{dt}{a} = \frac{1}{a} F(t) + c = \frac{1}{a} F(ax + b) + c$

F-2. $\int sin(ax + b)dx = -\frac{1}{a} cos(ax + b) + c$

Ex: $\int sin(2x + 4)dx = -\frac{1}{2} cos(2x + 4) + c$ $\{or\ we\ can\ substitute\ 2x + 4 = t$

Ex: $\int sin(2x)dx = -\frac{1}{2} cos(2x) + c$

Ex: $\int \sin(2-7x)\,dx = \frac{-cos(2-7x)}{-7} + c$

Ex: $\int sin\left(\frac{x}{3}\right)dx = -\frac{cos\frac{x}{3}}{\frac{1}{3}} + c = -3cos\left(\frac{x}{3}\right) + c$

F-3. $\int cos(ax+b)dx = \frac{1}{a}sin(ax+b) + c$

Ex: $\int cos4x\,dx = \frac{sin4x}{4} + c$

Ex: $\int \cos(2-x)\,dx = \frac{\sin(2-x)}{-1} + c$

F-4. $\int sec^2(ax+b)dx = \frac{1}{a}tan(ax+b) + c$

Ex: $\int sec^2 4x\,dx = \frac{tan4x}{4} + c$

Ex: $\int sec^2(2+x)dx = \tan(2+x) + c$

F-5. $\int cosec^2(ax+b)dx = -\frac{1}{a}cot(ax+b) + c$

Ex: $\int cosec^2 3x\,dx = -\frac{cot3x}{3} + c$

Ex: $\int cosec^2 7x\,dx = -$ cot7x/7+c

F-6. $\int sec(ax+b)tan(ax+b)dx = \frac{1}{a}sec(ax+b) + c$

Ex: $\int \sec(2x+1)\tan(2x+1)\,dx = \frac{\sec(2x+1)}{2} + c$

Ex: $\int \sec(x+1)\tan(x+1)\,dx = \sec(x+1) + c$

F-7. $\int cosec(ax+b)cot(ax+b)dx = -\frac{1}{a}cosec(ax+b) + c$

Ex: $\int \operatorname{cosec}(2x+1)\cot(2x+1)\,dx = -\frac{cosec(2x+1)}{2} + c$

Ex: $\int \operatorname{cosec}(x+1)\cot(x+1)\,dx = -cosec(x+1) + c$

F-8. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

Ex: $\int e^{2x+3}dx = \frac{e^{2x+3}}{2} + c$

Ex: $\int e^{-x}dx = \frac{e^{-x}}{-1} + c = -e^{-x} + c$

Ex: $\int e^{-2x}dx = \frac{e^{-2x}}{-2} + c$

F-9. $\int a^{bx+d}dx = \frac{1}{b}\frac{a^{bx+d}}{lna} + c$

F-10. $\int \frac{1}{ax+b}dx = \frac{1}{a}ln|ax+b| + c$

Ex: $\int \frac{1}{3x-2}dx = \frac{ln|3x-2|}{3} + c$

Ex: $\int \frac{1}{2-7x}dx = \frac{ln|2-7x|}{-7} + c$ (IMP)

Ex: $\int \frac{1}{2-x}dx = \frac{ln|2-x|}{-1} + c$

F-11. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c, n \neq -1$

Ex: $\int (2x+1)^{11}dx = \frac{1}{2}\frac{(2x+1)^{12}}{12} + c$

F-12. $\int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}tan^{-1}(ax+b) + c$

Ex: $\int \frac{1}{1+(2x+1)^2}dx = \frac{1}{2}tan^{-1}(2x+1) + c$

F-13. $\int \frac{1}{\sqrt{1-(ax+b)^2}}dx = \frac{1}{a}sin^{-1}(ax+b) + c$

Ex: $\int \frac{1}{\sqrt{1-(2x+1)^2}} = \frac{1}{2}sin^{-1}(2x+1) + c$

F-14. $\int \frac{1}{(ax+b)\sqrt{(ax+b)^2-1}}dx = \frac{1}{a}sec^{-1}(ax+b) + c$

F-15. $\int tanx\, dx = ln|secx| + c$

F-16. $\int cotx\, dx = ln|sinx| + c$

F-17. $\int secx\, dx = ln|secx + tanx| + c$

F-18. $\int cosecx\, dx = ln|cosecx - cotx| + c$

Ex: Evaluate $\int tan7x\, dx$

Ans: $\int tan7x\, dx$

$let\ 7x = t \Rightarrow 7\, dx = dt \Rightarrow dx = \frac{dt}{7}$

$=\int tant\ \frac{dt}{7} = \frac{1}{7}\int tant\, dt = \frac{1}{7}ln|sect| + c = \frac{1}{7}ln|sec7x| + c$

Ex: Evaluate $\int cot7x\, dx$

Ans: $\int cot7x\, dx$

$let\ 7x = t \Rightarrow 7\, dx = dt \Rightarrow dx = \frac{dt}{7}$

$=\int cott\ \frac{dt}{7} = \frac{1}{7}\int cott\, dt = \frac{1}{7}ln|sint| + c = \frac{1}{7}ln|sin7x| + c$

Ex: Evaluate $\int \sec(2x+1)\, dx$

Ans: $\int \sec(2x+1)\, dx$

$let\ 2x + 1 = t \Rightarrow 2\, dx = dt \Rightarrow dx = \frac{dt}{2}$

$=\int sect\ \frac{dt}{2} = \frac{1}{2}\int sect\, dt = \frac{1}{2}ln|sect + tant| + c$

$= \frac{1}{2}ln|\sec(2x+1) + \tan(2x+1)| + c$

Ex: Evaluate $\int \mathrm{cosec}(2x-3)\, dx$

Ans: $\int \mathrm{cosec}(2x-3)\, dx$

$let\ 2x - 3 = t \Rightarrow 2\, dx = dt \Rightarrow dx = \frac{dt}{2}$

$=\int cosect\ \frac{dt}{2} = \frac{1}{2}\int cosect\, dt = \frac{1}{2}ln|cosect - cott| + c$

$$=\frac{1}{2}ln|\text{cosec}(2x-3)-\cot(2x-3)|+c$$

IMP:

Ex: $\int \frac{sinx}{\sin(x-\alpha)}dx$

Ans: $\int \frac{sinx}{\sin(x-\alpha)}dx$

$let\ \ x-\alpha=t \Rightarrow dx=dt \qquad (x=t+\alpha)$

$=\int \frac{\sin(t+\alpha)}{\sin\ t}dt=\int \frac{\text{sint cos}\alpha+\text{cost sin}\alpha}{\text{sint}}dt=\int\left(\frac{\text{sint cos}\alpha}{\text{sint}}+\frac{\text{cost sin}\alpha}{sint}\right)dt$

$=\int cos\alpha\ dt+\int sin\alpha\ cott\ dt=cos\alpha\ \ t+sin\alpha\ \ ln|sint|+c$

$=cos\alpha\ (x-\alpha)+sin\alpha\ \ ln|\sin(x-\alpha)|+c$

Assignment:

Ex: $\int \frac{sinx}{\sin(x+\alpha)}dx$

Ex: $\int \frac{cosx}{\sin(x+\alpha)}dx$

Ex: $\int \frac{cosx}{\cos(x+\alpha)}dx$

Ex: $\int \frac{sinx}{\cos(x-\alpha)}dx$

Ex: $\int \tan(x+1)\,dx$

Ex: $\int \cot(2x-11)\,dx$

Ex: $\int \sec(3x)\,dx$

Ex: $\int \text{cosec}\ 7\text{x}\,dx$

Ex: $\int \sec^2(2x+1)\,dx$

Ex: $\int \sin(2x-11)\,dx$

Ex: $\int \sec(3x) tan3x\,dx$

Ex: $\int e^{2x-7}\,dx$

Ex: $\int \frac{1}{2-3x}\,dx$

IMP:

EX: $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

Ex: $\int \frac{tanx+tan\alpha}{tanx-tan\alpha}\,dx$

Ex: $\int cotx\sqrt{\ln sinx}\,dx$

Ex: $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$

Ex: $\int \frac{e^{tanx}}{cos^2x}\,dx$

Ex: $\int \frac{1}{sinx\,cosx}\,dx$

Ex: $\int \frac{1}{1-e^{-x}}\,dx$, $\int \frac{1}{1+e^{-x}}\,dx$, $\int \frac{1}{1+e^x}\,dx$

Ex: $\int \frac{sin2x}{acos^2x+bsin^2x}\,dx$

Ex: $\int \frac{cosec^2x}{3-cotx}\,dx$

Ex: $\int \frac{sinx}{cos^9x}\,dx$

Ex: $\int \frac{cosx}{\sqrt{1-sinx}}\,dx$

Ex: $\int \frac{x}{\sqrt{x^2-a^2}}\,dx$

Ex: $\int sinx\,e^{cosx}\,dx$

Ex: $\int x\sqrt{a^2+x^2}\,dx$

Ex: $\int cosx\,cos(sinx)dx \quad let\ sinx = t$

INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS (CH-3)

$$sin^2x = 1 - cos^2x\ , cos^2x = 1 - sin^2x\ , sec^2x = 1 + tan^2x\ ,$$

$$cosec^2x = 1 + cot^2x\ , cos^2x = \frac{1+cos2x}{2}, sin^2x = \frac{1-cos2x}{2}$$

$$Sin(A+B) + sin(A-B) = 2sinAcosB$$

$$Sin(A+B) - Sin(A-B) = 2cosAsinB$$

$$cos(A+B) + cos(A-B) = 2cosAcosB$$

$$\begin{cases} cos(A+B) - cos(A-B) = -2sinAsinB \\ or\ cos(A-B) - cos(A+B) = 2sinAsinB \end{cases}$$

Ex: $\int \boldsymbol{sin\ 3x\ cos2xdx}$

Ans: $\int sin\ 3x\ cos2xdx = \frac{1}{2}\int 2\ sin\ 3x\ cos2xdx$

$= \frac{1}{2}\int(\sin(3x+2x) + \sin(3x-2x))dx$

$= \frac{1}{2}\int sin5xdx + \frac{1}{2}\int sinx\ dx$

$= \frac{1}{2}\left(\frac{-cos5x}{5}\right) + \frac{1}{2}(-cosx) + c$

$= -\frac{cos5x}{10} - \frac{cosx}{2} + c$

Ex: $\int \boldsymbol{cos5x\ cos2xdx}$

Ans: $\int cos\ 5x\ cos2xdx = \frac{1}{2}\int 2\ cos5x\ cos2xdx$

$= \frac{1}{2}\int(\cos(5x+2x) + \cos(5x-2x))dx$

$= \frac{1}{2}\int cos7xdx + \frac{1}{2}\int cos3x\ dx$

$= \frac{1}{2}\left(\frac{sin7x}{7}\right) + \frac{1}{2}\left(\frac{sin3x}{3}\right) + c$

$= \frac{sin7x}{14} + \frac{sin3x}{6} + c$

Ex: $\int \boldsymbol{cos3x\ sinx\ sin5x\ dx}$

Ans: $\int cos3x\ sinx\ sin5xdx$

$= \frac{1}{2}\int (2sin5xcos3x)sinx\ dx$

$= \frac{1}{2}\int (sin8x + sin2x)sinxdx$

$= \frac{1}{2}\int sin8xsinx\ dx + \frac{1}{2}\int sin2xsinx\ dx$

$= \frac{1}{4}\int 2sin8x\ sinx\ dx + \frac{1}{4}\int 2sin2x\ sinx\ dx$

$$= \frac{1}{4}\int (\cos(8x - x) - \cos(8x + x))\,dx + \frac{1}{4}\int (\cos(2x - x) - \cos(2x + x))\,dx$$

$= \frac{1}{4}\int \cos 7x\,dx - \frac{1}{4}\int \cos 9x\,dx + \frac{1}{4}\int \cos x\,dx - \frac{1}{4}\int cos3xdx$

$= \frac{1}{4}\frac{sin7x}{7} - \frac{1}{4}\frac{sin9x}{9} + \frac{1}{4}sinx - \frac{1}{4}\frac{sin3x}{3} + c$

<u>Here the power of sinx and cosx are odd</u>

Ex: $\int \boldsymbol{sin^3x\,dx}$

Ans: $\int sin^3x\,dx = \int sin^2x\,sinxdx = \int (1 - cos^2x)sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$= \int (1 - t^2)(-dt) = \int (t^2 - 1)dt = \int t^2dt - \int dt = \frac{t^3}{3} - t + c$

$= \frac{cos^3x}{3} - cosx + c$

Ex: $\int \boldsymbol{sin^5x\,dx}$

Ans: $\int sin^5x\,dx = \int sin^4x\,sinxdx = \int (sin^2x)^2 sinxdx = \int (1 - cos^2x)^2 sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$=\int(1-t^2)^2(-dt) = \int(1+t^4-2t^2)(-dt) = \int(2t^2-t^4-1)dt$

$=\int 2t^2dt - \int t^4dt - \int dt = \frac{2t^3}{3} - \frac{t^5}{5} - t + c$

$=\frac{2cos^3x}{3} - \frac{cos^5x}{5} - cosx + c$

Ex: $\int sin^7x\,dx$

Ans: $\int sin^7x\,dx = \int sin^6x\,sinxdx = \int(sin^2x)^3sinxdx = \int(1-cos^2x)^3sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$=\int(1-t^2)^3(-dt) = \int(1-t^6-3t^2+3t^4)(-dt) = \int(t^6-3t^4+3t^2-1)dt$

$=\int t^6dt - \int 3t^4dt + \int 3t^2dt - \int dt = \frac{t^6}{6} - 3\frac{t^5}{5} + 3\frac{t^3}{3} - t + c$

$=\frac{cos^6x}{6} - 3\frac{cos^5x}{5} + 3\frac{cos^3x}{3} - cosx + c$ USE: $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

Ex: $\int sin^5x\,cos^3x$dx

Ans: $\int sin^5x\,cos^3xdx = \int sin^5x\,cos^2x\,cosxdx$

$= \int sin^5x\,(1-sin^2x)\,cosxdx$

Let $sinx = t \Rightarrow cosxdx = dt$

$=\int t^5\,(1-t^2)dt = \int t^5dt - \int t^7dt = \frac{t^6}{6} - \frac{t^8}{8} + c = \frac{sin^6x}{6} - \frac{sin^8x}{8} + c$

Ex: $\int sin^3x\,cos^8x$dx

Ans: $\int sin^3x\,cos^8xdx = \int cos^8x\,sin^3xdx = \int cos^8x\,sin^2x\,sinxdx$

$= \int cos^8x\,(1-cos^2x)\,sinxdx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$=\int t^8\,(1-t^2)(-dt) = \int(t^8-t^{10})(-dt)$

$= \int -t^8 dt + \int t^{10} dt = -\frac{t^9}{9} + \frac{t^{11}}{11} + c = -\frac{cos^9x}{9} + \frac{cos^{11}x}{11} + c$

Ex: $\int \frac{sin^3x}{cos^9x} dx$

Ans: $\int \frac{sin^3x}{cos^9x} dx = \int \frac{sin^2xsinx}{cos^9x} dx = \int \frac{(1-cos^2x)sinx}{cos^9x} dx$

Let $cosx = t \Rightarrow -sinxdx = dt$

$= \int \frac{(1-t^2)}{t^9}(-dt) = \int \frac{(t^2-1)}{t^9} dt = \int \frac{t^2}{t^9} dt - \int \frac{1}{t^9} dt = \int t^{-7} dt - \int t^{-9} dt$

$= \frac{t^{-6}}{-6} - \frac{t^{-8}}{-8} + c = -\frac{(cosx)^{-6}}{6} + \frac{(cosx)^{-9}}{9} + c$

Ex: $\int tan^2xdx$

Ans: $\int tan^2xdx = \int (sec^2x - 1)dx = \int sec^2xdx - \int 1\, dx = tanx - x + c$

Ex: $\int tan^4xdx$

Ans: $\int tan^4xdx = \int tan^2x\, tan^2xdx = \int tan^2x(sec^2x - 1)dx$

$= \int tan^2x\, sec^2xdx - \int tan^2xdx$

Let $tanx = t \Rightarrow sec^2xdx = dt$

$= \int t^2 dt - \int (sec^2x - 1)dx = \int t^2 dt - \int sec^2xdx + \int 1\, dx = \frac{t^3}{3} - tanx + x + c$

$= \frac{tan^3x}{3} - tanx + x + c$

Ex: $\int tan^6xdx$

Ans: $\int tan^6xdx = \int tan^4xtan^2xdx = \int tan^4x(sec^2x - 1)dx$

$= \int tan^4xsec^2xdx - \int tan^4xdx$

Let $tanx = t \Rightarrow sec^2xdx = dt$

$= \int t^4 dt - \int tan^2x(sec^2x - 1)dx$

$= \int t^4 dt - \int tan^2 x sec^2 x dx + \int tan^2 x\, dx$

$= \int t^4 dt - \int t^2 dt + \int (sec^2 x - 1) dx$

$= \int t^4 dt - \int t^2 dt + \int sec^2 x dx - \int 1\, dx$

$= \frac{t^5}{5} - \frac{t^3}{3} + tanx - x + c = \frac{tan^5 x}{5} - \frac{tan^3 x}{3} + tanx - x + c$

Ex: $\int cot^3 x dx$

Ans: $\int cot^3 x dx = \int cotx\, cot^2 x dx = \int cotx(cosec^2 x - 1) dx$

$= \int cotx\, cosec^2 x dx - \int cotx dx$

Let $cotx = t \Rightarrow -cosec^2 x dx = dt$

$= \int t(-dt) - \int cotx\, dx = -\frac{t^2}{2} - ln|sinx| + c$

$= -\frac{cot^2 x}{2} - ln|sinx| + c$

Note: The power of tan is either even or odd but the power of sec is even

Ex: $\int tan^5 x\, sec^2 x\, dx$

Ans: $\int tan^5 x\, sec^2 x\, dx$

Let $tanx = t \Rightarrow\ sec^2 x dx = dt$

$= \int t^5 dt = \frac{t^6}{6} + c = \frac{tan^6 x}{6} + c$

Ex: $\int tan^{10} x\, sec^4 x\, dx$

Ans: $\int tan^{10} x\, sec^4 x\, dx = \int tan^{10} x\, sec^2 x\ \ sec^2 x dx$

$= \int tan^{10} x\ \ (1 + tan^2 x) sec^2 x dx$

Let $tanx = t \Rightarrow\ sec^2 x dx = dt$

$=\int t^{10}(1+t^2)dt = \int t^{10}dt + \int t^{12}dt = \frac{t^{11}}{11} + \frac{t^{13}}{13} + c$

$= \frac{tan^{11}x}{11} + \frac{tan^{13}x}{13} + c$

Note: The power of sec is either odd or even but the power of tan is odd

Ex: $\int sec^{11}x\ tanx\ dx$

Ans: $\int sec^{11}x\ tanx\ dx = \int sec^{10}x\ secx\ tanx\ dx$

Let $secx = t \Rightarrow secx\ tanx\ dx = dt$

$= \int t^{10}dt = \frac{t^{11}}{11} + c = \frac{sec^{11}x}{11} + c$

Ex: $\int sec^{12}x\ tan^3x\ dx$

Ans: $\int sec^{12}x\ tan^3x\ dx = \int sec^{11}x\ tan^2x\ secx\ tanx\ dx$

$=\int sec^{11}x\ (sec^2x - 1)\ secx\ tanx\ dx$

Let $secx = t \Rightarrow secx\ tanx\ dx = dt$

$= \int t^{11}(t^2 - 1)dt = \int t^{13}dt - \int t^{11}dt = \frac{t^{14}}{14} - \frac{t^{12}}{12} + c$

$= \frac{sec^{14}x}{14} - \frac{sec^{12}x}{12} + c$

Ex: $\int sin^2x\ dx$

Ans: $\int sin^2x\ dx = \int \frac{1-cos2x}{2}dx = \int \frac{1}{2}dx - \int \frac{cos2x}{2}dx$

$=\frac{1}{2}\int dx - \frac{1}{2}\int cos2x\ dx = \frac{1}{2}x - \frac{1}{2}\frac{sin2x}{2} + c$

Ex: $\int cos^2x\, dx$

Ans: $\int cos^2x\, dx = \int \frac{1+cos2x}{2} dx = \int \frac{1}{2} dx + \int \frac{cos2x}{2} dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int cos2x\, dx = \frac{1}{2}x + \frac{1}{2}\frac{sin2x}{2} + c$

Ex: $\int sin^2 2x\, dx = \int \frac{1-cos4x}{2} dx = \int \frac{1}{2} dx - \int \frac{cos4x}{2} dx$ (Hints)

Ex: $\int cos^2 2x\, dx = \int \frac{1+cos4x}{2} dx = \int \frac{1}{2} dx + \int \frac{cos4x}{2} dx$ (Hints)

Ex: $\int sin^4x\, dx$

Ans: $\int sin^4x\, dx = \int (sin^2x)^2\, dx = \int \left(\frac{1-cos2x}{2}\right)^2 dx$

$= \frac{1}{4}\int (1 + cos^2 2x - 2cos2x) dx = \frac{1}{4}\int \left(1 + \frac{1+cos4x}{2} - 2cos2x\right) dx$

$= \frac{1}{4}\int \left(\frac{2+1+cos4x-4cos2x}{2}\right) dx = \frac{1}{8}\int (3 + cos4x - 4cos2x) dx$

$= \frac{1}{8}\int 3dx + \frac{1}{8}\int cos4x dx - \frac{1}{8}\int 4cos2x\, dx = \frac{3}{8}x + \frac{1}{8}\frac{sin4x}{4} - \frac{1}{2}\frac{sin2x}{2} + c$

Assignment:

Ex. $\int sin3x\, cos2x\, dx$

Ex. $\int cos5x\, sin3x\, dx$

Ex. $\int cos4x\, cos2x\, dx$

Ex. $\int sin3x\, sin4x\, dx$

Ex. $\int cos2x\, cos4x\, dx$

Ex. $\int sinx\, sin2x\, sin3x\, dx$

Ex. $\int cos^3x\, dx$

Ex. $\int cos^5x\, dx$

Ex.$\int cos^7x\, dx$

Ex.$\int sin^5xcos^2x\, dx$

Ex.$\int sin^4xcosx\, dx$

Ex.$\int sin^3xcos^2x\, dx$

Ex.$\int \frac{sin^3x}{cosx} dx$

Ex.$\int cosmx\, cosnx\, dx$

Ex.$\int \frac{cos4x-cos2x}{sin4x-sin2x} dx$

Ex.$\int sec^nx\, tanx\, dx$

Ex:$\int sin^5xcos^2x\, dx$

Ex:$\int sin^4xcosx\, dx$

Ex:$\int sin^3xcos^2x\, dx$

Ex:$\int sin^3xcos^3x\, dx$

Ex:$\int cot^2x\, dx$

Ex:$\int cot^4x\, dx$

Ex: $\int cot^7x\, dx$

Ex: $\int tan^3x\, dx$

Ex: $\int tan^5x\, dx$

Ex:$\int cot^{11}x\, cosec^2x\, dx$

Ex:$\int cot^{14}x\, cosec^4x\, dx$

Ex: $\int sec^{12}x\, tan^3x\, dx$

Ex: $\int cosec^{14}x\, cot^3x\, dx$

Ex: $\int cosec^{12}x\, cot^3x\, dx$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

Chapter-4

F-1:If the integral in the form of $\int \frac{dx}{\sqrt{a^2-x^2}}$ then substitute $x = a\sin\theta$, $\sin\theta = x/a$

Proof: $\int \frac{dx}{\sqrt{a^2-x^2}}$ put $x = a\sin\theta \Longrightarrow dx = a\cos\theta\, d\theta$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\frac{x}{a} + c$$

Formula: $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$

Ex: $\int \frac{dx}{\sqrt{9-x^2}}$

Ans: $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = sin^{-1}\frac{x}{3} + c$

Ex: $\int \frac{dx}{\sqrt{7-9x^2}}$

Ans: $\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{7}\right)^2-(3x)^2}}$

Let $3x = t \Longrightarrow 3dx = dt \Longrightarrow dx = dt/3$

$$= \int \frac{dt/3}{\sqrt{\left(\sqrt{7}\right)^2-t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{\left(\sqrt{7}\right)^2-t^2}} = \frac{1}{3}sin^{-1}\frac{t}{\sqrt{7}} + c = \frac{1}{3}sin^{-1}\frac{3x}{\sqrt{7}} + c$$

Ex: $\int \frac{xdx}{\sqrt{9-x^4}}$

Ans: $\int \frac{xdx}{\sqrt{9-x^4}} = \int \frac{xdx}{\sqrt{3^2-(x^2)^2}}$

Let $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = dt/2$

$=\int \frac{dt/2}{\sqrt{3^2-t^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{3^2-t^2}} = \frac{1}{2}sin^{-1}\frac{t}{3} + c = \frac{1}{2}sin^{-1}\frac{x^2}{3} + c$

Ex: $\int \frac{e^x dx}{\sqrt{9-e^{2x}}}$

Ans: $\int \frac{e^x dx}{\sqrt{9-e^{2x}}} = \int \frac{e^x dx}{\sqrt{3^2-(e^x)^2}}$

Let $e^x = t \Rightarrow e^x dx = dt$

$=\int \frac{dt}{\sqrt{3^2-t^2}} = sin^{-1}\frac{t}{3} + c = sin^{-1}\frac{e^x}{3} + c$

Ex: $\int \frac{cosx\, dx}{\sqrt{16-sin^2x}}$

Ans: $\int \frac{cosx\, dx}{\sqrt{16-sin^2x}} = \int \frac{cosx\, dx}{\sqrt{4^2-sin^2x}}$

Let $sinx = t \Rightarrow cosx\, dx = dt$

$=\int \frac{dt}{\sqrt{4^2-t^2}} = sin^{-1}\frac{t}{4} + c = sin^{-1}\frac{sinx}{4} + c$

Ex: $\int \frac{dx}{x\sqrt{25-(lnx)^2}}$

Ans: $\int \frac{dx}{x\sqrt{25-(lnx)^2}}$

$= \int \frac{dx}{x\sqrt{5^2-(lnx)^2}}$

Let $lnx = t \Rightarrow \frac{1}{x}dx = dt$

$=\int \frac{dt}{\sqrt{5^2-t^2}} = sin^{-1}\frac{t}{5} + c = sin^{-1}\frac{lnx}{5} + c$

F-2: If the integral in the form of $\int \frac{dx}{a^2+x^2}$ then substitute $x = a\tan\theta, \tan\theta = x/a$

Proof: $\int \frac{dx}{a^2+x^2}$

Put $x = a\tan\theta \Longrightarrow dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Formula: $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + c$

Ex: $\int \frac{dx}{9+x^2}$

Ans: $\int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2} = \frac{1}{3}tan^{-1}\frac{x}{3} + c$

Ex: $\int \frac{dx}{11+16x^2}$

Ans: $\int \frac{dx}{11+16x^2} = \int \frac{dx}{(\sqrt{11})^2+(4x)^2}$

Let $4x = t \Longrightarrow 4\,dx = dt \Longrightarrow dx = dt/4$

$$= \int \frac{dt/4}{(\sqrt{11})^2+(t)^2} = \frac{1}{4}\int \frac{dt}{(\sqrt{11})^2+(t)^2} = \frac{1}{4}\frac{1}{\sqrt{11}}tan^{-1}\frac{t}{\sqrt{11}} + c = \frac{1}{4\sqrt{11}}tan^{-1}\frac{4x}{\sqrt{11}} + c$$

Ex: $\int \frac{x^2}{16+x^6}dx$

Ans: $\int \frac{x^2}{16+x^6}dx = \int \frac{x^2}{4^2+(x^3)^2}dx$

Let $x^3 = t \Longrightarrow 3x^2 dx = dt \Longrightarrow x^2 dx = dt/3$

$$= \int \frac{dt/3}{4^2+(t)^2} = \frac{1}{3}\int \frac{dt}{4^2+(t)^2} = \frac{1}{3}\frac{1}{4}tan^{-1}\frac{t}{4} + c = \frac{1}{12}tan^{-1}\frac{x^3}{4} + c$$

Ex: $\int \frac{e^{2x}dx}{9+e^{4x}}$

Ans: $\int \frac{e^{2x}dx}{9+e^{4x}} = \int \frac{e^{2x}dx}{3^2+(e^{2x})^2}$

Let $e^{2x} = t \Rightarrow e^{2x}\,2dx = dt \Rightarrow e^{2x}\,dx = dt/2$

$=\int \frac{dt/2}{3^2+t^2} = \frac{1}{2}\int \frac{dt}{3^2+t^2} = \frac{1}{2}\frac{1}{3}tan^{-1}\frac{t}{3} + c = \frac{1}{6}tan^{-1}\frac{e^{2x}}{3} + c$

Ex: $\int \frac{sec^2x\,dx}{16+tan^2x}$

Ans: $\int \frac{sec^2xdx}{16+tan^2x} = \int \frac{sec^2xdx}{4^2+tan^2x}$

Let $tanx = t \Rightarrow sec^2x\,dx = dt$

$=\int \frac{dt}{4^2+t^2} = \frac{1}{4}tan^{-1}\frac{tanx}{4} + c$

Ex: $\int \frac{dx}{x(25+(lnx)^2)}$

Ans: $\int \frac{dx}{x(25+(lnx)^2)} = \int \frac{dx}{x(5^2+(lnx)^2)}$

Let $lnx = t \Rightarrow \frac{1}{x}\,dx = dt$

$=\int \frac{dt}{5^2+t^2} = \frac{1}{5}tan^{-1}\frac{t}{5} + c = \frac{1}{5}tan^{-1}\frac{lnx}{5} + c$

F-3: If the integral in the form of $\int \frac{dx}{\sqrt{a^2+x^2}}$ then substitute $x = a\tan\theta$

Proof: $\int \frac{dx}{\sqrt{x^2+a^2}}$

$$\text{Put } x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$$

$$= \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \frac{a\sec^2\theta d\theta}{a\sqrt{\tan^2\theta + 1}} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\frac{x}{a} + \sqrt{\tan^2\theta + 1}\right| + c_1 = \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right| + c_1 = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 + a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c_1 = \ln\left|x + \sqrt{x^2 + a^2}\right| - \ln a + c_1 = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{9+x^2}}$

Ans: $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{dx}{\sqrt{3^2+x^2}} = ln\left|x + \sqrt{3^2 + x^2}\right| + c = ln\left|x + \sqrt{9 + x^2}\right| + c$

Ex: $\int \frac{dx}{\sqrt{7+9x^2}}$

Ans: $\int \frac{dx}{\sqrt{7+9x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{7}\right)^2+(3x)^2}}$

Let $3x = t \Rightarrow 3dx = dt \Rightarrow dx = dt/3$

$$= \int \frac{dt/3}{\sqrt{\left(\sqrt{7}\right)^2+t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{\left(\sqrt{7}\right)^2+t^2}} = \frac{1}{3} ln\left|t + \sqrt{\left(\sqrt{7}\right)^2 + t^2}\right| + c$$

$$= \frac{1}{3} ln\left|3x + \sqrt{7 + 9x^2}\right| + c$$

Ex: $\int \frac{x^4 dx}{\sqrt{9+x^{10}}}$

Ans: $\int \frac{x^4 dx}{\sqrt{9+x^{10}}} = \int \frac{x^4 dx}{\sqrt{3^2+(x^5)^2}}$

Let $x^5 = t \Rightarrow 5x^4 dx = dt \Rightarrow x^4 dx = dt/5$

$$= \int \frac{dt/5}{\sqrt{3^2+t^2}} = \frac{1}{5}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{5} ln\left|t + \sqrt{(3)^2 + t^2}\right| + c = \frac{1}{5} ln\left|x^5 + \sqrt{9 + x^{10}}\right| + c$$

Ex: $\int \frac{a^x dx}{\sqrt{9+a^{2x}}}$

Ans: $\int \frac{a^x dx}{\sqrt{9+a^{2x}}} = \int \frac{a^x dx}{\sqrt{3^2+(a^x)^2}}$

Let $a^x = t \Rightarrow a^x \ln a \, dx = dt \Rightarrow a^x \, dx = dt/\ln a$

$$=\int \frac{\frac{dt}{\ln a}}{\sqrt{3^2+t^2}} = \frac{1}{\ln a}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{\ln a} \ln\left|t + \sqrt{(3)^2+t^2}\right| + c$$

$$= \frac{1}{\ln a} \ln\left|a^x + \sqrt{9+a^{2x}}\right| + c$$

Ex: $\int \frac{cosec^2 x \, dx}{\sqrt{16+cot^2 x}}$

Ans: $\int \frac{cosec^2 x \, dx}{\sqrt{16+cot^2 x}} = \int \frac{cosec^2 x dx}{\sqrt{4^2+cot^2 x}}$

Let $cotx = t \Rightarrow -cosec^2 x dx = dt \Rightarrow cosec^2 x dx = -dt$

$$=\int \frac{-dt}{\sqrt{4^2+t^2}} = -\ln\left|t + \sqrt{(4)^2+t^2}\right| + c = -\ln\left|cotx + \sqrt{16+cot^2 x}\right| + c$$

Ex: $\int \frac{dx}{x\sqrt{25+(\ln x)^2}}$

Ans: $\int \frac{dx}{x\sqrt{25+(\ln x)^2}}$

$$= \int \frac{dx}{x\sqrt{5^2+(\ln x)^2}}$$

Let $\ln x = t \Rightarrow \frac{1}{x} dx = dt$

$$=\int \frac{dt}{\sqrt{5^2+t^2}} = \ln\left|t + \sqrt{(5)^2+t^2}\right| + c = \ln\left|\ln x + \sqrt{25+(\ln x)^2}\right| + c$$

F-4: If an integral in the form of $\int \frac{dx}{\sqrt{x^2-a^2}}$ then substitute $x = a\sec\theta$

Proof: $\int \frac{dx}{\sqrt{x^2-a^2}}$

$$\text{Put } x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$$

$$= \int \frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sqrt{\sec^2\theta - 1}} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\frac{x}{a} + \sqrt{\sec^2\theta - 1}\right| + c_1 = \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right| + c_1 = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1 = \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + c_1 = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{x^2-9}}$

Ans: $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}} = ln\left|x + \sqrt{x^2 - 3^2}\right| + c = ln\left|x + \sqrt{x^2 - 9}\right| + c$

Ex: $\int \frac{dx}{\sqrt{9x^2-7}}$

Ans: $\int \frac{dx}{\sqrt{9x^2-7}} = \int \frac{dx}{\sqrt{(3x)^2-\left(\sqrt{7}\right)^2}}$

Let $3x = t \Longrightarrow 3dx = dt \Longrightarrow dx = dt/3$

$$= \int \frac{dt/3}{\sqrt{t^2-\left(\sqrt{7}\right)^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{t^2-\left(\sqrt{7}\right)^2}} = \frac{1}{3} ln\left|t + \sqrt{t^2 - \left(\sqrt{7}\right)^2}\right| + c$$

$$= \frac{1}{3} ln\left|3x + \sqrt{9x^2 - 7}\right| + c$$

Ex: $\int \frac{x^3 dx}{\sqrt{x^8-9}}$

Ans: $\int \frac{x^3 dx}{\sqrt{x^8-9}} = \int \frac{x^3 dx}{\sqrt{(x^4)^2-3^2}}$

Let $x^4 = t \Rightarrow 4x^3 \mathrm{dx} = \mathrm{dt} \Rightarrow x^3 \mathrm{dx} = \mathrm{dt}/4$

$= \int \frac{dt/4}{\sqrt{t^2-3^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{t^2-3^2}} = \frac{1}{4} ln\left|t + \sqrt{t^2 - 3^2}\right| + c = \frac{1}{4} ln\left|x^4 + \sqrt{x^8 - 9}\right| + c$

Ex: $\int \frac{e^x dx}{\sqrt{e^{2x}-11}}$

Ans: $\int \frac{e^x dx}{\sqrt{e^{2x}-11}} = \int \frac{e^x dx}{\sqrt{(e^x)^2-\left(\sqrt{11}\right)^2}}$

Let $e^x = t \Rightarrow \mathrm{e^x\ dx} = \mathrm{dt}$

$= \int \frac{dt}{\sqrt{t^2-\left(\sqrt{11}\right)^2}} = ln\left|t + \sqrt{t^2 - \left(\sqrt{11}\right)^2}\right| + c$

$= ln\left|e^x + \sqrt{e^{2x} - 11}\right| + c$

Ex: $\int \frac{dx}{sin^2x\sqrt{cot^2x-16}}$

Ans: $\int \frac{dx}{sin^2x\sqrt{cot^2x-16}} = \int \frac{cosec^2x dx}{\sqrt{cot^2x-4^2}}$

Let $cotx = t \Rightarrow -cosec^2xdx = \mathrm{dt} \Rightarrow cosec^2xdx = -\mathrm{dt}$

$= \int \frac{-dt}{\sqrt{t^2-4^2}} = -ln\left|t + \sqrt{t^2 - 4^2}\right| + c = -ln\left|cotx + \sqrt{cot^2x - 16}\right| + c$

Ex: $\int \frac{dx}{x\sqrt{(lnx)^2-25}}$

Ans: $\int \frac{dx}{x\sqrt{(lnx)^2-25}}$

$= \int \frac{dx}{x\sqrt{(lnx)^2-5^2}}$

Let $lnx = t \Rightarrow \frac{1}{\mathrm{x}} \mathrm{dx} = \mathrm{dt}$

$=\int \frac{dt}{\sqrt{t^2-5^2}} = ln\left|t+\sqrt{t^2-5^2}\right| + c = ln\left|lnx + \sqrt{(lnx)^2-25}\right| + c$

IMPORTANT FORMS:

If the integral in the form $\int \frac{ax \pm b}{\sqrt{a^2-x^2}}dx$ or $\int \frac{ax \pm b}{\sqrt{a^2+x^2}}dx$ or $\int \frac{ax \pm b}{\sqrt{x^2-a^2}}dx$ or $\int \frac{ax \pm b}{a^2+x^2}dx$

then express $\int \frac{ax \pm b}{\sqrt{a^2-x^2}}dx = \int \frac{ax}{\sqrt{a^2-x^2}}dx \pm \int \frac{b}{\sqrt{a^2-x^2}}dx$

Let $I_1 = \int \frac{ax}{\sqrt{a^2-x^2}}dx$ and $I_2 = \int \frac{b}{\sqrt{a^2-x^2}}$

Integrate I_1 by putting $a^2-x^2=t^2$ or $a^2-x^2=t$

Integrate I_2 by using the above formula .

Similarly we can integrate other forms.

Ex: $\int \frac{3x-2}{\sqrt{9-x^2}}dx$

Ans: $\int \frac{3x-2}{\sqrt{9-x^2}}dx = \int \frac{3x}{\sqrt{9-x^2}}dx - \int \frac{2}{\sqrt{9-x^2}}dx$

$= 3\int \frac{x}{\sqrt{9-x^2}}dx - 2\int \frac{dx}{\sqrt{3^2-x^2}}$

In the 1st term let $9-x^2=t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$

$= 3\int \frac{-tdt}{t} - 2\int \frac{dx}{\sqrt{3^2-x^2}} = -3t - 2sin^{-1}\frac{x}{3} + c = -3\sqrt{9-x^2} - 2sin^{-1}\frac{x}{3} + c$

Ex: $\int \frac{x+2}{\sqrt{16+x^2}}dx$

Ans: $\int \frac{x+2}{\sqrt{16+x^2}}dx = \int \frac{x}{\sqrt{16+x^2}}dx + \int \frac{2}{\sqrt{16+x^2}}dx$

In the 1st term let $16+x^2=t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

$= \int \frac{tdt}{t} + 2\int \frac{dx}{\sqrt{4^2+x^2}} = t + 2\,ln\left|x+\sqrt{16+x^2}\right| + c$

$=\sqrt{16+x^2} + 2\,ln\left|x+\sqrt{16+x^2}\right| + c$

Ex: $\int \frac{2x+5}{7+x^2} dx$

Ans: $\int \frac{2x+5}{7+x^2} dx = \int \frac{2x}{7+x^2} dx + \int \frac{5}{7+x^2} dx$

Let $7 + x^2 = t \Rightarrow 2xdx = dt$

$= \int \frac{dt}{t} + 5 \int \frac{dx}{(\sqrt{7})^2 + x^2} = ln|t| + 5 \frac{1}{\sqrt{7}} tan^{-1} \frac{x}{\sqrt{7}} + c$

$= ln|7 + x^2| + 5 \frac{1}{\sqrt{7}} tan^{-1} \frac{x}{\sqrt{7}} + c$

Ex: $\int \frac{3x+1}{\sqrt{x^2-25}} dx$

Ans: $\int \frac{3x+1}{\sqrt{x^2-25}} dx = \int \frac{3x}{\sqrt{x^2-25}} dx + \int \frac{1}{\sqrt{x^2-25}} dx$

$= 3 \int \frac{x}{\sqrt{x^2-25}} dx + \int \frac{1}{\sqrt{x^2-5^2}} dx$

In the 1st term let $x^2 - 25 = t^2 \Rightarrow 2xdx = 2tdt \Rightarrow xdx = tdt$

$= 3 \int \frac{tdt}{t} + \int \frac{1}{\sqrt{x^2-5^2}} dx = 3\, t + ln\left|x + \sqrt{x^2 - 5^2}\right| + c$

$= 3\sqrt{x^2 - 25} + ln\left|x + \sqrt{x^2 - 25}\right| + c$

IMP:

If the integral in the form $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

then express $ax^2 + bx + c = a\{(x - \alpha)^2 \pm \beta^2\}$ (In a perfect suare)

put $x - \alpha = t \Rightarrow dx = dt$

Ex: $\int \frac{dx}{x^2+4x+9}$

Ans: $\int \frac{dx}{x^2+4x+9}$

Make the denominator $x^2 + 4x + 9$ in a perfect square

Now $x^2 + 4x + 9 = x^2 + 2.x.2 + 2^2 - 2^2 + 9 = (x+2)^2 + 5$

$:So \int \frac{dx}{x^2+4x+9} = \int \frac{dx}{(x+2)^2+5} = \int \frac{dx}{(x+2)^2+(\sqrt{5})^2}$

Let $x + 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$$= \int \frac{dt}{t^2+(\sqrt{5})^2} = \frac{1}{\sqrt{5}} tan^{-1} \frac{t}{\sqrt{5}} + c = \frac{1}{\sqrt{5}} tan^{-1} \frac{x+2}{\sqrt{5}} + c$$

Ex:$\int \frac{dx}{\sqrt{x^2-4x+13}}$

Ans: $\int \frac{dx}{\sqrt{x^2-4x+13}}$

Make the denominator $x^2 - 4x + 13$ in a perfect square

Now $x^2 - 4x + 13 = x^2 - 2.x.2 + 2^2 - 2^2 + 13 = (x-2)^2 + 9$

$:So \int \frac{dx}{\sqrt{x^2-4x+13}} = \int \frac{dx}{\sqrt{(x-2)^2+9}} = \int \frac{dx}{\sqrt{(x-2)^2+3^2}}$

Let $x - 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$$= \int \frac{dt}{\sqrt{t^2+3^2}} = ln\left|t + \sqrt{t^2+3^2}\right| + c = ln\left|(x-2) + \sqrt{(x-2)^2 + 3^2}\right| + c$$

Ex:$\int \frac{dx}{\sqrt{x^2-4x-13}}$

Ans: $\int \frac{dx}{\sqrt{x^2-4x-13}}$

Now $x^2 - 4x - 13 = x^2 - 2.x.2 + 2^2 - 2^2 - 13 = (x-2)^2 - 17$

$:So \int \frac{dx}{\sqrt{x^2-4x-13}} = \int \frac{dx}{\sqrt{(x-2)^2-17}} = \int \frac{dx}{\sqrt{(x-2)^2-(\sqrt{17})^2}}$

Let $x - 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$$= \int \frac{dt}{\sqrt{t^2-(\sqrt{17})^2}} = ln\left|t + \sqrt{t^2 - (\sqrt{17})^2}\right| + c$$

$$= ln\left|(x-2) + \sqrt{(x-2)^2 - (\sqrt{17})^2}\right| + c$$

Ex: $\int \frac{dx}{\sqrt{5-4x-x^2}}$

Ans: $\int \frac{dx}{\sqrt{5-4x-x^2}}$

Make the denominator $5 - 4x - x^2$ in a perfect square

Now $5 - 4x - x^2 = -(x^2 + 4x - 5) = -(x^2 + 2.x.2 + 2^2 - 2^2 - 5)$

$$= -((x+2)^2 - 9) = 9 - (x+2)^2$$

:So $\int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}} = \int \frac{dx}{\sqrt{3^2-(x+2)^2}}$

Let $x + 2 = t \Rightarrow \mathrm{dx} = \mathrm{dt}$

$= \int \frac{dt}{\sqrt{3^2-t^2}} = sin^{-1}\frac{t}{3} + c = sin^{-1}\frac{x+2}{3} + c$

If the integral in the form $\int \frac{(\mathrm{px}+\mathrm{q})\,\mathrm{dx}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}$ or $\int \frac{(\mathrm{px}+\mathrm{q})\mathrm{dx}}{\sqrt{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}}$

then express $\mathrm{px}+\mathrm{q} = \mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}) + \mathrm{m}$
,compare the coefficient of x and constant term , find the value of l and m.

Now the given integration can be written in the form of $\int \frac{(\mathrm{px}+\mathrm{q})\,\mathrm{dx}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}$

$$= \int \frac{\mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}) + \mathrm{m}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}\mathrm{dx} = \int \frac{\mathrm{l}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}^2+\mathrm{bx}+\mathrm{c})}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}} + \int \frac{\mathrm{m}}{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}\mathrm{dx}$$

Solve it by using the formula.

Ex: $\int \frac{(3x+2)dx}{x^2+4x+9}$

Ans: $\int \frac{(3x+2)dx}{x^2+4x+9}$

Let $3x + 2 = l\frac{d}{dx}(x^2 + 4x + 9) + m$

$$= l(2x + 4) + m$$

$$= 2lx + 4l + m$$

Compare the coefficient of x and constant term in both the sides

$$2l = 3 \Rightarrow l = \frac{3}{2} \quad and\ \ 4l + m = 2 \Rightarrow m = 2 - 4l = 2 - 4\,.\frac{3}{2} = -4$$

$$So\ \ \int \frac{(3x+2)dx}{x^2+4x+9} = \int \frac{l(2x+4)+m}{x^2+4x+9}dx = \int \frac{l(2x+4)}{x^2+4x+9}dx + \int \frac{m}{x^2+4x+9}dx$$

$$= l\int \frac{(2x+4)}{x^2+4x+9}dx + m\int \frac{dx}{x^2+4x+9}$$

Let $x^2 + 4x + 9 = t \Rightarrow (2x + 4)dx = dt$

$$= l\int \frac{dt}{t} + m\int \frac{dx}{x^2+2.x.2+2^2-2^2+9} = l\ ln|t| + m\int \frac{dx}{(x+2)^2+5}$$

Let $x + 2 = z \Rightarrow dx = dz$

$$= l\ ln|t| + m\int \frac{dz}{(z)^2+\left(\sqrt{5}\right)^2} = \frac{3}{2}ln|x^2 + 4x + 9| - 4tan^{-1}\frac{z}{\sqrt{5}} + c$$

$$= \frac{3}{2}ln|x^2 + 4x + 9| - 4tan^{-1}\frac{x+2}{\sqrt{5}} + c$$

Assignment:

Ex: $\int \frac{dx}{4+9x^2}$

Ex: $\int \frac{3x}{1+2x^4}$

Ex: $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}} dx$

Ex: $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}} dx$

Ex: $\int \frac{2^x}{\sqrt{9+4^x}} dx$

Ex: $\int \frac{dx}{\sqrt{16x^2+25}}$

Ex: $\int \frac{1}{\sqrt{a^2+b^2x^2}} dx$

Ex: $\int \frac{x^2}{\sqrt{x^6-1}} dx$

Ex: $\int \frac{e^{-x}}{16+9e^{-2x}} dx$

Ex: $\int \frac{1}{e^x+e^{-x}} dx$

Ex: $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

Ex: $\int \frac{5x-2}{\sqrt{11-x^2}} dx$

Ex: $\int \frac{3x+5}{\sqrt{x^2-9}} dx$

Ex: $\int \frac{3x-2}{16+x^2} dx$

Ex: $\int \frac{dx}{9x^2-12x+8}$

Ex: $\int \frac{dx}{2x^2+x+3}$

Ex: $\int \frac{2x+3}{\sqrt{5-4x-x^2}} dx$

Ex: $\int \frac{x-2}{\sqrt{4+x^2}} dx$

Ex: $\int \frac{dx}{\sqrt{2-4x+x^2}}$

Ex: $\int \frac{1}{\sqrt{7-6x-x^2}} dx$

INTEGRATION BY PARTS

If u and v are two functions then integration of the product of u and v is defined as

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

Here the 1st function can be choosed by using a form **ILATE**.

I-Inverse function $(sin^{-1}x, cos^{-1}x, tan^{-1}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

L-Logarithm function $(logx, \log(x+1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

A-algebraic function$(x, x^2, (x+1)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

T-Trigonometric function$(sinx, cosx, tanx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

E-Exponential function $(e^x, a^x, e^{x+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.)$

Ex: $\int x\,e^x\,dx$ here u = x and v= e^x

Ans: $\int x\,e^x\,dx = x\int e^x\,dx - \int\left\{\frac{d(x)}{dx}\int e^x dx\right\}dx$

$= x\,e^x - \int 1\,e^x dx = xe^x - e^x + c$

Ex: $\int x\ sinx\ dx$

Ans: $\int x\ sinx\ dx = x\int sinx\, dx - \int\left\{\frac{d(x)}{dx}\int sinx\ dx\right\}dx$

$= x\,(-cosx) - \int 1\,(-cosx)\,dx = -x\ cosx + sinx + c$

Ex: $\int x\ cosx\ dx = x\int cosx\, dx - \int\left\{\frac{d(x)}{dx}\int cosx\ dx\right\}dx$

$= x\,(sinx) - \int 1\,(sinx)\,dx = x\ sinx + cosx + c$

Ex: $\int x\ e^{2x}\ dx$

Ans: $\int x\ e^{2x}\ dx = x\int e^{2x}\,dx - \int\left\{\frac{d(x)}{dx}\int e^{2x}dx\right\}dx$

$= x\ \frac{e^{2x}}{2} - \int 1\ \frac{e^{2x}}{2}\ dx = x\ \frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x}dx + c = x\ \frac{e^{2x}}{2} - \frac{1}{2}\ \frac{e^{2x}}{2} + c$

Ex: $\int x\ sin3x\ dx$

Ans: $\int x\ sin3x\ dx = x\int sin3x\ dx - \int\left\{\frac{d(x)}{dx}\int sin3x\ dx\right\}dx$

$= x\left(\frac{-cos3x}{3}\right) - \int 1\left(\frac{-cos3x}{3}\right)dx = -x\frac{cos3x}{3} + \frac{1}{3}\int cos3x\ dx + c$

$= -x\frac{cos3x}{3} + \frac{1}{3}\frac{sin3x}{3} + c$

Ex: $\int x\ sec^2x\ dx$

Ans: $\int x\ sec^2x\ \ dx = x\int sec^2x\,dx - \int\left\{\frac{d(x)}{dx}\int sec^2x\ dx\right\}dx$

$= x\ tanx\ - \int 1\ tanx\ dx = x\ tanx - ln|secx| + c$

Ex: $\int (x+1)\ e^x\ dx$

Ans: $\int (x+1)\ e^x\ dx = (x+1)\int e^x\,dx - \int\left\{\frac{d(x+1)}{dx}\int e^x dx\right\}dx$

$= (x+1)e^x - \int 1\ e^x dx = (x+1)e^x - e^x + c$

OR

$\int (x+1)\, e^x\, dx = \int x\, e^x dx + \int e^x dx$

Ex: $\int x\, tan^2 x\, dx$

Ans: $\int x\, tan^2 x\ \ dx = \int x\, (\, sec^2 x - 1) dx$

$= \int x\, sec^2 x\ \ dx - \int\ x\, dx$

$= x \int sec^2 x\, dx - \int \left\{ \frac{d(x)}{dx} \int sec^2 x\, dx \right\} dx - \frac{x^2}{2}$

$= x\, tanx\ - \int 1\, tanx\, dx - \frac{x^2}{2} = x\, tanx - ln|secx| - \frac{x^2}{2} + c$

Ex: $\int x\, cos^2 x\, dx$

Ans: $\int x\, cos^2 x\ \ dx = \int x\, \left(\frac{1+cos2x}{2}\right) dx = \int \frac{x}{2} dx + \int \frac{x\, cos2x}{2}\ dx$

$= \frac{1}{2} \int x\, dx + \frac{1}{2} \int x\, cos2x\, dx$

$= \frac{1}{2}\, \frac{x^2}{2} + \frac{1}{2} \left\{ x \int cos2x\, dx - \int \left\{ \int \frac{d(x)}{dx} \int cos2x\, dx \right\} dx \right\}$

$= \frac{x^2}{4} + \frac{1}{2} \left\{ x\, \left(\frac{sin2x}{2}\right) - \int 1\, \left(\frac{sin2x}{2}\right)\, dx \right\}$

$= \frac{x^2}{4} + \frac{1}{2} \left\{ x\, \left(\frac{sin2x}{2}\right) - \frac{1}{2} \int\ sin2x\, dx \right\}$

$= \frac{x^2}{4} + \frac{1}{2} \left\{ x\, \left(\frac{sin2x}{2}\right) - \frac{1}{2} \left(\frac{-cos2x}{2}\right) \right\} + c$

Ex: $\int x\, sin3x\, cos2x\, dx$

Ans: $\int x\, sin3x\, cos2x\, dx = \frac{1}{2} \int x\, (2sin3x\, cos2x)\, dx$

$= \frac{1}{2} \int x\, (\sin(3x+2x) + \sin\,(3x-2x))\, dx$

$= \frac{1}{2} \int x\, (\sin 5x + \sin x) dx = \frac{1}{2} \int x\ \ sin5x\, dx + \frac{1}{2} \int x \sin x\, dx$

Ex: $\int lnx\, dx$

Ans: $\int lnx\, dx = lnx \int 1\, dx - \int \left\{\frac{d(lnx)}{dx} \int 1\, dx\right\} dx$

$= lnx\, x - \int \frac{1}{x}\, x\, dx = x\, lnx - \int dx = x\, lnx - x + c$

Ex: $\int x^5\, lnx\, dx$

Ans: $\int x^5\, lnx\, dx = lnx \int x^5\, dx - \int \left\{\frac{d(lnx)}{dx} \int x^5\, dx\right\} dx$

$= lnx\, \frac{x^6}{6} - \int \frac{1}{x} \frac{x^6}{6}\, dx = \frac{x^6}{6} lnx - \frac{1}{6} \int x^5 dx = \frac{x^6}{6} lnx - \frac{1}{6} \frac{x^6}{6} + c$

$= \frac{x^6}{6} lnx - \frac{x^6}{36} + c$

Ex: $\int x \ln(1+x)\, dx$

Ans: $\int x \ln(1+x)\, dx = \ln(x+1) \int x\, dx - \left\{\frac{d(\ln(x+1))}{dx} \int x\, dx\right\} dx$

$= \ln(x+1)\, \frac{x^2}{2} - \int \frac{1}{x+1} \frac{x^2}{2} dx = x \ln(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} dx$

$= x \ln(x+1) - \frac{1}{2} \int \frac{x^2-1+1}{x+1} dx$

$= x \ln(x+1) - \frac{1}{2} \left\{\int \frac{x^2-1}{x+1} dx + \int \frac{1}{x+1} dx\right\}$

$= x \ln(x+1) - \frac{1}{2} \left\{\int \frac{(x+1)(x-1)}{x+1} dx + \int \frac{1}{x+1} dx\right\}$

$= x \ln(x+1) - \frac{1}{2} \int (x-1) dx - \frac{1}{2} \int \frac{1}{x+1} dx$

$= x \ln(x+1) - \frac{1}{2} \int x\, dx + \frac{1}{2} \int dx - \frac{1}{2} \int \frac{1}{x+1} dx$

$= x \ln(x+1) - \frac{1}{2} \frac{x^2}{2} + \frac{1}{2}\, x - \frac{1}{2} ln|x+1| + c$

IMP

Ex: $\int \ln(1+x^2)\,dx$

Ans: $\int \ln(1+x^2)\,dx = \ln(1+x^2)\int 1\,dx - \int\left\{\frac{d}{dx} ln(1+x^2)\int 1\,dx\right\}dx$

$= \ln(1+x^2)\,x - \int \frac{1}{1+x^2}\,2x\,x\,dx = x\,ln(1+x^2) - 2\int \frac{x^2}{1+x^2}dx$

$= x\,ln(1+x^2) - 2\int \frac{x^2+1-1}{1+x^2}dx = x\,ln(1+x^2) - 2\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$

$= x\,ln(1+x^2) - 2\int dx + 2\int \frac{1}{1+x^2}dx = x\,ln(1+x^2) - 2x + 2\,tan^{-1}x + c$

Ex: $\int sin^{-1}x\,dx$

Ans: $\int sin^{-1}x\,dx = sin^{-1}x\ \int 1\,dx - \int\left\{\frac{d}{dx} sin^{-1}x \int 1\,dx\right\}dx$

$= sin^{-1}x\ .x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx = x.sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

For 2nd term let $1-x^2 = t^2 \ \Rightarrow\ -2x\,dx = 2t\,dt \ \Rightarrow x\,dx = -t\,dt$

$= x.sin^{-1}x - \int \frac{-t\,dt}{t}\ = x.sin^{-1}x + t + c = x.sin^{-1}x + \sqrt{1-x^2} + c$

Ex: $\int tan^{-1}x\,dx$

Ans: $\int tan^{-1}x\,dx = tan^{-1}x\ \int 1\,dx - \int\left\{\frac{d}{dx} tan^{-1}x \int 1\,dx\right\}dx$

$= tan^{-1}x\ .x - \int \frac{1}{1+x^2}\,x\,dx = x.tan^{-1}x - \int \frac{x\,dx}{1+x^2}$

For 2nd term let $1+x^2 = t \ \Rightarrow\ 2x\,dx = \ dt \ \Rightarrow x\,dx = dt/2$

$= x.tan^{-1}x - \int \frac{dt/2}{t}\ = x.tan^{-1}x + \frac{1}{2}ln|t| + c$

$= x.tan^{-1}x + \frac{1}{2}ln|1+x^2| + c$

Ex: $\int x\, tan^{-1}x\, dx$

Ans: $\int x\, tan^{-1}x\, dx = tan^{-1}x \int x\, dx - \int \left\{\frac{d}{dx} tan^{-1}x \int x\, dx\right\} dx$

$= tan^{-1}x\ .\frac{x^2}{2} - \int \frac{1}{1+x^2}\ .\frac{x^2}{2}\ dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int \frac{x^2\ dx}{1+x^2}$

$= \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\left\{\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right\} dx$

$= \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2} dx = \frac{x^2}{2}\ tan^{-1}x - \frac{1}{2}\ x + \frac{1}{2} tan^{-1}x + c$

Assignment:

Ex: $\int x\, sinx\, cosx\, dx$

Ex: $\int x\, sin5x\, dx$

Ex: $\int xe^{bx} dx$

Ex: $\int xcos^2x\, dx$

Ex: $\int xcosnx\, dx$

Ex: $\int x\, lnx\, dx$

Ex: $\int \frac{lnx}{x^5}\ dx = \int x^{-5} lnx\, dx$

Ex: $\int x^n\ lnx\, dx$

Ex: $\int cos^{-1}x\, dx$

Ex: $\int (lnx)^2 dx$